J Autom Reasoning (2014) 53:407–429
DOI 10.1007/s10817-014-9311-3

On the Formalization of Gamma Function in HOL

Umair Siddique · Osman Hasan

Received: 25 November 2011 / Accepted: 11 July 2014 / Published online: 25 July 2014

Abstract The Gamma function is a special transcendental function that is widely used in probability theory, fractional calculus and analytical number theory. This paper presents a higher-order logic formalization of the Gamma function using the HOL4 theorem prover. The contribution of this paper can be mainly divided into two parts. Firstly, we extend the existing integration theory of HOL4 by formalizing a variant of improper integrals using sequential limits. Secondly, we build upon these results to formalize the Gamma function and verify some of its main properties, such as pseudo-recurrence relation, functional equation and factorial generalization. In order to illustrate the practical effectiveness and utilization of our work, we formally verify some properties of Euler's generalized power rule of differentiation, Mittag-Leffler functions and the relationship between the Exponential and Gamma random variables.

Keywords Gamma function · Mittag-Leffler function · Fractional calculus · Gamma random variable

1 Introduction

The most common method for computing the factorial of an integer number m, i.e., $m!$, is by recursively multiplying all the integers from 1 to m. However, this method becomes inefficient as the value of m increases. To address this problem, Euler proposed an integral based formula for $m!$ in 1729, which was later upgraded by Legendre into the commonly

U. Siddique (✉) · O. Hasan
School of Electrical Engineering and Computer Science (SEECS), National University of Sciences and Technology (NUST), Islamabad, Pakistan
e-mail: umair.siddique@seecs.nust.edu.pk

O. Hasan
e-mail: osman.hasan@seecs.nust.edu.pk

known Gamma function [49]. One of the most important characteristics of the Gamma function is its ability to generalize the factorial over non-integer numbers. The Gamma function is widely used in many areas of Mathematics (e.g. fractional calculus [9], analytical number theory [43] and probability theory [50]) and Physics (e.g. quantum mechanics and fluid dynamics [32]). These foundational concepts are in turn used to model and analyze many engineering systems, including electronic components [10], electromagnetic systems [12] and wireless sensor networks [2].

The most commonly used expression of the Gamma function, for a real number $z > 0$, is given by an improper integral as follows:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{1}$$

The usage of improper integral in the above definition allows us to deal with situations when the integrand becomes unbounded in the given interval or the interval itself becomes unbounded.

In the past couple of decades, formal verification of safety-critical engineering systems has become a dire need. The rigorous modeling and verification methods using in formal verification, usually increases the chances for catching subtle but critical errors that are often ignored by traditional techniques, like numerical computations and paper-and-pencil based proof methods [46]. Model checking [28], which is an automatic formal verification method, has been widely used to analyze many applications. However, due to the continuous nature of the analysis and the involvement of transcendental functions, it cannot be used for the formal analysis of the Gamma function based applications. On the other hand, we believe that higher-order-logic theorem proving [13] offers a promising solution for conducting formal analysis of such applications. The main reason being the highly expressive nature of higher-order logic, which can be leveraged upon to essentially model any system that can be expressed in a closed mathematical form. In fact, most of the classical mathematical theories behind elementary calculus, such as limits, differentiation and integration have already been formalized in higher-order logic [17]. However, to the best of our knowledge, formal reasoning support for the Gamma function is not reported in the literature so far.

In this paper, we build upon the available theories of elementary calculus and extend them to formalize a variant of improper integrals (we call it sequential improper integral) and then the Gamma function using the HOL4 theorem prover [47]. The main motivations of using the HOL4 theorem prover include the availability of Harrison's seminal work on the formalization of elementary calculus [17], which we build upon, and the availability of probabilistic analysis framework [20], which we intend to extend by the Gamma distribution [50], using the current work. In this paper, besides the formal definition of the Gamma function (Γ) and the lower γ_l and upper Γ_u incomplete Gamma functions, we also present the formal verification of some of their classical properties: Pseudo-recurrence relation ($\Gamma(z+1) = z\Gamma(z)$), Functional equation ($\Gamma(1) = 1$), Factorial generalization ($\Gamma(k+1) = k!$), Reconstruction of the Gamma Function ($\Gamma(z) = \Gamma_u(a, z) + \gamma_l(a, z)$), Recurrence Relation of Lower Incomplete Gamma Function ($\gamma_l(a, z+1) = z\gamma_l(a, z - a^z)exp^{(-a)}$) and the Recurrence Relation of Upper Incomplete Gamma Function ($\Gamma_u(a, z) = (z-1)\Gamma_u(a, (z-1)) + a^{(z-1)}exp^{(-s)}$). These results not only ensure the correctness of our formal definitions of the Gamma function but also could play a vital role in the formal analysis of many applications involving the Gamma function.

In this paper, we also present three applications of the Gamma function, i.e., formalization of Euler's generalized rule of differentiation, Mittag-Leffler functions and the relationship between the Exponential and Gamma random variables. Euler's generalized rule of differentiation allows us to take the derivatives of non-integer orders. While the Mittag-Leffeler functions generalize the Exponential and Hyperbolic functions. Both of these mathematical results are quite frequently used in the design of fractional order systems [10]. Finally, the Gamma random variable is a widely used continuous random variable and the formal verification of its relationship with the Exponential random variable plays an important role in extending the capability of formal probabilistic analysis using theorem proving [21].

The rest of the paper is organized as follows: Section 2 describes some related work about the Gamma function computation and the formalization of integral calculus. Section 3 provides a brief introduction to the HOL4 theorem prover and presents an overview of Harrison's formalization of derivatives and the Gauge integral. Section 4 presents an introduction to improper integrals and their formalization in the HOL theorem prover. The formalization of the Gamma function and the verification of its key properties are presented in Section 5. In order to demonstrate the practical effectiveness and the utilization of our work, we present three applications in Section 6. Finally, Section 7 concludes the paper and highlights some future directions.

2 Related Work

Traditionally, the analysis of the Gamma function based applications has been done using paper-and-pencil proof methods. However, considering the complexity of present age engineering and scientific systems, such kind of analysis is notoriously difficult, if not impossible, and is quite error prone. Many examples of erroneous paper-and-pencil based proofs are available in the open literature, a recent one can be found in [8] and its identification and correction is reported in [41]. One of the most commonly used computer based analysis technique for the Gamma function is numerical computations [31] that cannot provide accurate results as well due to the involvement of infinite summations and huge memory requirements. Similarly, the computation of the Gamma function $\Gamma(x)$ for large values of x is not possible in such numerical computation software packages. For example, MATLAB [36] returns `7.26e306` as the approximated value computed for $x = 171$ and returns `Inf` for all values beyond $x = 171$. Another alternative to analyze the Gamma function is computer algebra systems [4], which are very efficient for computing mathematical solutions symbolically, but they are not reliable [17] due to their limitations of dealing with side conditions. Another limitation of computer algebra systems is the uncertain simplification of singular expressions (where the argument of gamma function can result in an undefined value, for example $\Gamma(0) = \infty$) [1]. Another source of inaccuracy in computer algebra systems is the presence of unverified huge symbolic manipulation algorithms in their core, which are quite likely to contain bugs. Thus, these traditional techniques should not be relied upon for the analysis of applications involving the computation of the Gamma function, especially when they are used in safety-critical areas (e.g., cardiac tissue electrode interface [34]), where inaccuracies may even result in the loss of human lives.

The early formalization of main concepts of Calculus in higher-order logic, such as, limits, derivatives and integrals were laid down by Harrison [17]. Butler [5] reported the formalization of integral Calculus in the PVS theorem prover. L. Cruz-Filipe reported a constructive theory of analysis in the Coq theorem prover in his Ph.D dissertation [7]. Mhamdi [37] presented the higher-order logic formalization of Lebesgue integration theory in the HOL4 theorem prover, which is a fundamental concept in many mathematical theories. In this paper, we are providing a framework that can be used to formalize improper integrals in the HOL4 theorem prover. Our work uses and extends the work done by Harrison [18]. The main reasons behind this choice include the richness of Harrison's real analysis related theories, which are fundamental to our work, and the ability to use Harrison's Gauge integral to formalize the notion of improper integrals and thus the Gamma function.

3 Preliminaries

In this section, we provide a brief introduction to the HOL4 theorem prover and present an overview of Harrison's [17] formalization of derivatives and the Gauge integral. The intent is to introduce the basic theories along with some notations that are going to be used in the rest of the paper.

3.1 HOL Theorem Prover

HOL is an interactive theorem proving environment for the construction of mathematical proofs in higher-order logic. The first version of HOL was developed by Mike Gordon at Cambridge University, in 1980's. The core of HOL is interfaced to the functional programming language ML - Meta Language [42]. HOL utilizes the simple type theory of Church [6] along with Hindley-Milner polymorphism [39] to implement higher-order logic. The first version of HOL was called HOL88 and the later ones include HOL90 and HOL98. HOL4 is the recent version of the HOL family and it uses Moscow ML which is an implementation of the Standard ML (SML). The HOL core consists of only 5 basic axioms and 8 primitive inference rules, which are implemented as ML functions. HOL has been widely used for the formal verification of software and hardware systems along with the formalization of mathematical theories.

HOL supports four types of terms: constants, variables, function applications, and lambda-terms. Variables are sequences of digits or letters beginning with a letter, e.g., y, b, Gamma_hol. Applications in HOL represent the evaluation of a function at an argument. HOL uses λ-terms, also called lambda abstractions, for denoting functions. For example, the lambda abstraction function `λx.f(x)` represents a function which takes x and returns $f(x)$. According to the lambda calculus implemented in HOL, every HOL term has a unique type which is either one of the basic types or the result of applying a type constructor to other types. When a term is entered into HOL, the type is inferred using the type checking algorithm implemented in HOL. If the type of a term cannot be deduced automatically then it is advisable to explicitly mention it, e.g., (`x : real`) and (`x : bool`) assign the types `real` and `bool` to the variable `x`, respectively.

A theorem is a formalized statement that may be an axiom or could be deduced from already verified theorems by applying inference rules. A theorem consists of a finite set of boolean terms Ω called the assumptions and a boolean term S called the conclusion. For example, if (Ω, S) is a theorem in HOL then it is written as $\Omega \vdash S$. A HOL theory consists of a set of types, type operators, constants, definitions, axioms and theorems. HOL theories are organized in a hierarchical fashion and theories can have other theories as parents through which they can inherit all of their types, constants, definitions and theorems. We utilized the HOL theories of Booleans, positive integers, real numbers, sequences, limits and transcendental functions in our work. In fact, one of the primary motivations of selecting the HOL4 theorem prover for our work was to benefit from these built-in mathematical theories.

In HOL, there are two types of interactive proof methods : forward and backward. In a forward proof, the user starts from the primitive inference rules and tries to prove the goals on top of these rules and already proved theorems. The forward proof method is not an easy approach as it requires all the low level details of the proof in advance. A backward or a goal directed proof method is the reverse of the forward proof method. It is based on the concept of a tactic; which is an ML function that breaks goals into simple subgoals. In the goal directed proof method, the user starts with the desired theorem or the main goal that is further reduced to simpler subgoals using the *tactics*. There are many automatic proof procedures and proof assistants [15] available in HOL which help the user in directing the proof to the end. In interactive theorem proving, the user interacts with the HOL proof editor and guides the prover by manually applying the necessary tactics until the given goal is verified.

Table 1 provides the mathematical interpretations of some frequently used HOL symbols and functions in this paper.

3.2 Harrison's Formalization of Integer order Calculus

In this section, we present a brief introduction to the existing higher-order-logic formalization of the integer order derivative and the Gauge integral. The notations and definitions of this section will be useful for the reader in the next sections in which we present our formalization of the improper integrals and Gamma function.

Table 1 HOL symbols and functions

HOL Symbol	Standard Symbol	Meaning
$\wedge$	and	Logical *and*
$\vee$	or	Logical *or*
$\sim$	not	Logical *negation*
$\Rightarrow$	$\longrightarrow$	Implication
$\Leftrightarrow$	$=$	Equality
`@x.t`	$\epsilon x.t$	an x such that : t
`λx.t`	$\lambda x.t$	Function that maps x to $t(x)$
`num`	$\{0, 1, 2, \ldots\}$	Positive Integers data type
`real`	All Real numbers	Real data type
`suc n`	$(n + 1)$	Successor of natural number
`ln x`	$log_e(x)$	Natural logarithm function
`exp x`	e^x	Exponential function
`sqrt x`	$\sqrt{x}$	Square root function
`abs x`	$\|x\|$	Absolute function
`lim(λn.f(n))`	$\lim_{n\to\infty} f(n)$	Limit of a *real* sequence f
`convergent(λn.f(n))`	$\exists x. \lim_{n\to\infty} f(n) = x$	f is convergent
`suminf(λn.f(n))`	$\lim_{k\to\infty} \sum_{n=0}^{k} f(n)$	Infinite summation of f
`summable(λn.f(n))`	$\exists x. \lim_{k\to\infty} \sum_{n=0}^{k} f(n) = x$	Summation of f is convergent
`((λx.f(x)) → l)y`	$\lim_{x\to y} f(x) = l$	Limit of f as x approaches y

3.2.1 Formalization of Derivative

Harrison [17] formalized the *real number theory* along with the fundamentals of Calculus, such as real sequences, summation series, limits of a function and derivatives and verified most of their classical properties in HOL. Some of the important definitions related to derivative are given as follows:

Definition 1 Continuity of a Function
```
⊢ ∀ f x. f contl x = ((λ h. f (x+h)) → f x) (0)
```

Definition 2 Derivative of a Function
```
⊢ ∀ f l x.(f diffl l) x ⇔ ((λh.(f (x+h) - f x)/h) → l) (0)
```

Definition 3 Differentiability of a Function
```
⊢ ∀ f x. f differentiable x = ∃l. (f diffl l) (x)
```

Where Definition 1 defines the continuity of a function f at point x, i.e., $f(x+h)$ approaches $f(x)$ as h approaches 0. The Definition 2 gives the derivative of a function f at point x, i.e., the limit value of $\frac{f(x+h)-f(x)}{h}$ as h approaches 0. Finally, Definition 3 formalizes the condition of the differentiability of a function.

3.2.2 Gauge Integral

Various integrals have been proposed in literature, namely, the Newton integral, the Riemann integral and the Lebesgue integral, and each one of them has its own advantages and disadvantages. For example, the Newton integral may not exist for certain functions, e.g., the step function, the Riemann integral does not have convenient convergence properties and the Lebesgue integral, which is better than Riemann integral, shares one problem with the Riemann integral that the Fundamental Theorem of Calculus is not always true [17].

In the 1960's, Kurzweil and Henstock proposed a new integral that is simple and powerful and it incorporates every function the others integrals can integrate. It is normally known as the *Generalized Riemann Integral*, the *Kurzweil-Henstock Integral* or the *Gauge Integral*. The main features of the Gauge integral are that it has all the convergence properties of Lebesgue integral and it generalizes the Riemann integral. Particularly, in case of real functions, the Gauge integral is capable of integrating more functions than the Lebesgue integral.

The formalization of the Gauge integral [17] requires three important predicates, namely, (`division`) which is used to test whether a function represents a division or not, `tdiv` which represents the tagged division and `dsize`, which is used to extract the size of division. The HOL formalization of these predicates is given as follows:

Definition 4 Interval Division
```
⊢ ∀ a b D. division (a,b) D =
  (D 0 = a) ∧ ∃N.(∀ n. n < N ⟹ D n < D(SUC n)) ∧ ∀n. n ≥ N
      ⟹ (D n = b)
```

Definition 5 Interval Tagged Division

```
⊢ ∀ a b D p. tdiv (a,b) (D,p) =
        division (a,b) D ∧ ∀n. D n ≤ p n ∧ p n ≤ D (SUC n)
```

The above predicates are defined for checking whether a function represents a division or tagged division of an interval. A predicate that extracts the size of the division is defined as follows [17]::

Definition 6 Size of Division

```
⊢ ∀ D. dsize D =
     @N.(∀ n. n < N ⟹ D n < D (SUC n)) ∧ ∀n. n ≥ N
       ⟹ (D n = D N)
```

Finally, the notion of a function f being gauge over the closed interval E and the concept of fine tagged division with respect to a gauge is defined as::

Definition 7 Gauge

```
⊢ ∀ E f. gauge E f = ∀x. E x ⟹ 0 < f x
```

Definition 8 Fine Tagged Division

```
⊢ ∀ g D p. fine g (D,p) =
     ∀n. n < dsize D ⟹ D (SUC n) - D n < g (p n)
```

The above formalization can now be used to define Riemann sum over tagged division as follows [17]::

Definition 9 Riemann Sum

```
⊢ ∀ D p f. rsum (D,p) f =
     sum (0,dsize D) (λn. f (p n) * (D (SUC n) - D n))
```

Now, the definite integration, i.e., Dint (a, b) f k $\Longleftrightarrow \int_a^b f(x)\,dx = K$, can be defined as:

Definition 10 Definite Integral

```
⊢ ∀ a b f k. Dint (a,b) f k =
    ∀ e.
      0 < e ⇒
        ∃ g. gauge (λ x. a ≤ x ∧ x ≤ b) g ∧
      ∀ D p. tdiv (a,b) (D,p) ∧
          fine g (D,p) ⟹ abs (rsum (D,p) f - k) < e
```

The definition of an integral and integrable function is given as follows:

Definition 11 Integral

```
⊢ ∀ a b f . integral (a,b) f = @k. Dint (a,b) f k
```

Definition 12 Integrable Function

```
⊢ ∀ a b f . integrable (a,b) f = ∃ k. Dint (a,b) f k
```

Where @ is the Hilbert choice operator in HOL. Currently, only a few properties of the Gauge integral are available in the HOL4 theorem prover [47] distribution compared to the ones that are available in the HOL-Light [16] theorem prover. Since these additional properties play a vital role in reasoning about the improper integrals, so we ported them to the HOL4 theorem prover as part of the reported work. Some of the important properties of the Gauge integral, that we ported, are given in Table 2. As mentioned in Section 1 as well, the reason behind using the HOL4 theorem prover rather than HOL-Light for our work is to be able to utilize the formalized Gamma function along with the formal probabilistic analysis framework [20] to formalize the Gamma distribution, which is widely used to model many physical phenomena in engineering systems.

4 Formalization of Sequential Improper Integrals

Improper integrals are primarily used either when the integrand becomes unbounded in the given interval of integration or the interval is itself unbounded. As an example of the first case, consider a function f that is defined on an interval [a,∞). For every value $t > a$, the

Table 2 Properties of Gauge integral

Property	HOL Formalization
DINT_INTEGRAL	`⊢ ∀ f g a b. (a ≤ b) ∧ (Dint (a,b) f g) ⟹ (integral (a,b) f = g)`
INTEGRAL_LINEAR	`⊢ ∀ f a b c.(a ≤ b) ∧ (integrable(a,b) f) ∧ (integrable(a,b) g) ⟹ integral(a,b) (m*f + n*g) = m*integral(a,b) f + n*integral(a,b) g)`
INTEGRAL_BY_PARTS	`⊢ ∀ f g f′ g′ a b.(a ≤ b)∧ (∀ x. a ≤ x ∧ x ≤ b ⟹ (f diffl f′(x))(x))∧ (∀ x. a ≤ x ∧ x ≤ b ⟹ (g diffl g′(x))(x))∧ integrable(a,b) (λx. f′( x * g x)∧ integrable(a,b) (λx. f x * g′(x) ⟹ (integral(a,b) (λx. f x * g′ x) = (f b * g b - f a * g a) - integral(a,b) (λx. f′(x) * g x))`
INTEGRABLE_CONTINUOUS	`⊢ ∀ f a b. (∀ x. a ≤ x ∧ x ≤ b ⇒ f contl x) ⟹ integrable(a,b) f`
INTEGRAL_LE	`⊢ ∀ f g a b.(a ≤ b) ∧ (integrable(a,b) f)∧ (integrable(a,b) g) ∧ (∀ x. a ≤ x ∧ x ≤ b ⇒ f(x) ≤ g(x)) ⟹ integral(a,b) f ≤ integral(a,b) g`
INTEGRAL_COMBINE	`⊢ ∀ f a b c.(a ≤ b) ∧ (integrable(a,b) f) ∧ (integrable(b,c) f) ⟹ integral(a,c) f = integral(a,b) f + integral(b,c) f)`
INTEGRABLE_SUBINTERVAL	`⊢ ∀ f a b c d. (a ≤ c) ∧ (c ≤ d) ∧ (d ≤ b) ∧ integrable(a,b) f ⟹ integrable(c,d) f`

function is integrable on [a,t] and if the limit $\lim_{t\to\infty}\int_a^t f(x)dx$, exists, then the theory of improper integrals allows us to write:

$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx \tag{2}$$

Similarly, if the limit $\lim_{t\to-\infty}\int_t^b f(x)dx$ exists, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx \tag{3}$$

As an example of the second case, consider a function f that is bounded in the closed interval [t,b] for every $a < t < b$ but is unbounded, or undefined, on a. Now, if the limit $\lim_{t\to a+}\int_t^b f(x)dx$, exists, then the theory of improper integrals allows us to write:

$$\int_a^{b} f(x)dx = \lim_{t\to a+}\int_t^b f(x)dx \tag{4}$$

Similarly, if the limit $\lim_{t\to b-}\int_a^t f(x)dx$ exists, then

$$\int_a^{b} f(x)dx = \lim_{t\to b-}\int_a^t f(x)dx \tag{5}$$

In a similar manner, many other interesting cases for the utilization of improper integrals can be deduced from different combinations of (2) to (5).

Generally, the improper integrals are defined using the real limits (i.e., the function over which the limit is taken has the form $f : \mathbb{R} \to \mathbb{R}$). However, in order to build upon the existing formalization of HOL4 theorem prover, and thus minimize the formalization effort, we utilize sequential limits to formalize a variant of improper integrals, which we refer to as the sequential improper integral. This variant of improper integral suffices for the formalization of the Gamma functions, which is the main focus of the current paper. Moreover, it is very useful in the context of evaluating the improper integral of monotonic functions. We formalize our sequential improper integral to handle different cases as follows:

Definition 13 Sequential Improper Integral
```
⊢ ∀ g h f. seq_improper_integral (g,h) f =
          lim (λn. lim(λm. integral (g n,h m) f))
```

Where `f` is a function of type $\mathbb{R} \to \mathbb{R}$ that represents the integrand. The functions `g` and `h` are of type $\mathbb{N} \to \mathbb{R}$ and represent the lower and upper limits of integration. Thus, by assigning appropriate values to functions `g` and `h`, we can model different types of improper integrals, as illustrated in Table 3.

Note that the last entry in the table is for functions that exhibit special behaviours for integer-valued arguments. For example, the function $sin(2\pi x)$ returns a 0 for all integer values for the parameter x and its improper integral should not exist due to continuous fluctuations in this function across the whole real-line. Such kind of functions can be dealt with our definition by using a multiplication factor a, which does not have an integer equivalent, in the functions g and h of our definition as indicated in the last row of Table 3. Thus, the improper integral of the function $sin(2\pi x)$ would not exist, which is the expected behaviour.

Table 3 Modeling of some special cases of improper integrals

Cases	Modeling with seq_improper_Integral
$\int_{a+}^{b} f(x)dx$	seq_improper_Integral (λn. a + $\frac{1}{2^n}$, λm. b) f
$\int_{a}^{b-} f(x)dx$	seq_improper_Integral (λn. a, λm. b − $\frac{1}{2^m}$) f
$\int_{a+}^{b-} f(x)dx$	seq_improper_Integral (λn. a + $\frac{1}{2^n}$, λm. b − $\frac{1}{2^m}$) f
$\int_{a+}^{\infty} f(x)dx$	seq_improper_Integral (λn. a + $\frac{1}{2^n}$, λm. m) f
$\int_{-\infty}^{b} f(x)dx$	seq_improper_Integral (λn. − n, λm. b) f
$\int_{a}^{\infty} f(x)dx$	seq_improper_Integral (λn. a, λm. m) f
$\int_{-\infty}^{\infty} f(x)dx$	seq_improper_Integral (λn. − n, λm. m) f
$\int_{-\infty}^{\infty} f(x)dx$ where f is a function with special behavior on integer values e.g., sin, cos etc.	seq_improper_Integral ((λn. − a ∗ &n), (λm.a ∗ &m)) f where a is a real number

Next, we present the formal verification of two key properties (4) and (5) of improper integrals.

Theorem 1 *Lower-Plus Improper Integral Equivalence*

⊢ ∀ f a b g. (∀x. a ≤ x ∧ x ≤ b ⇒ f contl x) ∧ (a < b) ⟹
   (integral (a,b) f = seq_improper_Integral (λn. a + $\frac{1}{2^n}$, λm. b) f)

The assumptions guarantee the continuity of the function f in the interval [a, b], where b is always greater than a.

Proof Sketch We start the proof process by rewriting the right-hand-side with the definition of improper integral (Definition 13), which results in the following subgoal:

$$\text{integral (a, b) f} = \text{lim } (\lambda \text{n. integral } (\text{a} + \frac{1}{2^{\text{n}}}, \text{b) f}) \tag{6}$$

Now, rewriting with the definition of limit results in the following two subgoals:

$$(\exists \text{x. } (\lambda \text{n. integral } (\text{a} + \frac{1}{2^{\text{n}}}, \text{b) f}) \to \text{ x}) \tag{7}$$

$$\forall \text{x. } (\lambda \text{n. integral } (\text{a} + \frac{1}{2^{\text{n}}}, \text{b) f}) \to \text{ x} \Longrightarrow (\text{integral (a, b) f } = \text{ x}) \tag{8}$$

In the first subgoal, we specialize the value of x with integral (a, b) f and then using some of the integral properties, given in Table 1, along with some arithmetic reasoning, we reach the following subgoal:

$$(\lambda \text{n.integral (a, b) f} - \text{integral } (\text{a, a} + \frac{1}{2^{\text{n}}}) \text{ f}) \longrightarrow \text{integral (a, b) f} \tag{9}$$

The verification of above subgoal mainly requires the following lemmas.

Lemma 1 *Squeeze Theorem*

⊢ ∀ f g h a. (∀n. (p n ≤ g n) ∧ (g n ≤ h n)) ∧
   (p ⟶ a) ∧ (h ⟶ a) ⟹ (g ⟶ a) □

Lemma 2 *Bounds of an Integral*

$\vdash \forall$ f a b. ($\forall$x. a $\leq$ x $\wedge$ x $\leq$ b $\Rightarrow$ f contl x) $\Longrightarrow$
((λn. inf (IMAGE f (λx. a $\leq$ x $\wedge$ x $\leq$ b))) * (b-a)) $\leq$
integral (a,b) f $\wedge$
integral (a,b) f $\leq$
((λn. sup (IMAGE f (λx. a $\leq$ x $\wedge$ x $\leq$ b))) * (b-a)

Lemma 3 *Limits of Infimum and Supremum of a Continuous Function in* $[g\ n, b]$

$\vdash \forall$ f a b c. ($\forall$x. a $\leq$ x $\wedge$ x $\leq$ b $\Rightarrow$ f contl x)$\wedge$ g $\rightarrow a$ $\wedge$
($\forall$n. a < g n) $\wedge$ (g n < b) $\Longrightarrow$
((λn. inf (IMAGE f (λx. g n $\leq$ x $\wedge$ x $\leq$ b))) $\longrightarrow$ f a) $\wedge$
((λn. sup (IMAGE f (λx. g n $\leq$ x $\wedge$ x $\leq$ b))) $\longrightarrow$ f a)

Lemma 4 *Limit of an Integral of a Continuous Function in the interval* $[a, gn]$

$\vdash \forall$ f a b c. ($\forall$x. a $\leq$ x $\wedge$ x $\leq$ b $\Rightarrow$ f contl x) $\wedge$ (g $\rightarrow$ a)
$\Longrightarrow$ ((λn. integral (a, g n) f) $\longrightarrow$ 0)

Lemma 1 is verified using properties of limit of a real sequence. Lemma 2 presents the bounds of an integral and Lemma 3 shows that the limit of supremum (upper bound) and infimum (lower bound) of the function f in the interval [g n, b] as n becomes very large. Here, g n is used to model a+ as it approaches to a as n becomes very large. The verification of Lemmas 2 and 3 is mainly based on the properties of Gauge integral and set theory principles along with some arithmetic reasoning. Lemma 4 can be simplified using the Modus Ponens rule with Lemma 1 to get the following two subgoals:

$$\exists \mathtt{p}.\ (\forall \mathtt{n}.\ \mathtt{p\ n} \leq (\lambda \mathtt{n}.\ \mathtt{integral\ (a, g\ n)\ f})\ \mathtt{n} \wedge (\lambda \mathtt{n}.\ \mathtt{p\ n}) \longrightarrow 0 \tag{10}$$

$$\exists \mathtt{h}.\ (\forall \mathtt{n}.\ (\lambda \mathtt{n}.\ \mathtt{integral\ (a, g\ n)\ f})\ \mathtt{n} \leq \mathtt{h\ n} \wedge (\lambda \mathtt{n}.\ \mathtt{h\ n}) \longrightarrow 0 \tag{11}$$

We verified these subgoals by utilizing Lemmas 2 and 3, and specializing p and h with the following values :

$$\mathtt{p\ n} = (\lambda \mathtt{n}.\ \mathtt{inf\ (IMAGE\ f}\ (\lambda \mathtt{x}.\ \mathtt{a} \leq \mathtt{x} \wedge \mathtt{x} \leq \mathtt{g\ n})) * (\mathtt{g\ n} - \mathtt{a})$$

$$\mathtt{h\ n} = (\lambda \mathtt{n}.\ \mathtt{sup\ (IMAGE\ f}\ (\lambda \mathtt{x}.\ \mathtt{a} \leq \mathtt{x} \wedge \mathtt{x} \leq \mathtt{g\ n})) * (\mathtt{g\ n} - \mathtt{a})$$

Now the remaining subgoal (8) of Theorem 1 is verified based on the uniqueness of limit of a real sequence along with some arithmetic reasoning.

The second important property related to improper integrals, given in (5), is formalized in HOL as follows:

Theorem 2 *Upper-Minus Improper Integral Equivalence*

$\vdash \forall$ f a b. ($\forall$x. a $\leq$ x $\wedge$ x $\leq$ b $\Rightarrow$ f contl x) $\wedge$ (a < b) $\Longrightarrow$
(integral(a,b) f = seq_improper_Integral (λn. a, λm. b $- \frac{1}{2^m}$) f)

The proof steps for this theorem are quite similar to the ones for Theorem 1.

Besides the above mentioned key properties of improper integrals, we have also verified many other interesting cases of improper integrals and the details can be found in our HOL script [44]. One worth mentioning formally verified theorem in our development is the Improper Integral Theorem, according to which, if the improper integral of a function f exists in the interval $(s, b]$ for all $a < s$ then the integral of f in the interval $[a, b]$ also exists. We formalized this theorem in HOL as follows:

Theorem 3 *Improper Integral Theorem*
$\vdash \forall$ f a b.
(∃y. ∀x. a ≤ x ∧ x ≤ b ⇒ abs (f x) ≤ y) ∧ (a ≤ b) ∧
(∀n. ∃k. Dint($a + \frac{(b-a)}{2^n}$,b) f k) ∧
(∃l. λn. integral($a + \frac{(b-a)}{2^n}$,b) f → l) $\Longrightarrow$
(∃z. Dint(a,b) f = z)

The first two assumptions ensure that the function f is bounded within the interval $[a, b]$ and the next two assumptions guarantee that the improper integral $\lim_{s\to a^+} \int_s^b f(x)dx$ exists. The proof of Theorem 3 was quite tedious since it involves a significant amount of arithmetic reasoning along with the basic definitions of the Gauge integral and the improper integral and some of their associated properties.

Another interesting result of our work is the formal proof of the improper integral shift theorem as follows:

Theorem 4 *Improper Integral Shift Theorem*
$\vdash \forall$ f a. (∀x.f differentiable x) ∧
convergent (λm. integral (0, m) f) ∧
convergent (λn. integral (−n, 0) f) $\Longrightarrow$
seq_improper_Integral ((λn. − n), (λm. m)) f =
seq_improper_Integral ((λn. − n), (λm. m)) (λx. f (x + a))

The first assumption ensures the differentiability of the function f. The second and third assumptions ensure the convergence of f, i.e., the the limit exists at positive and negative infinity. The proof of this theorem is quite long and it is mainly based on the properties of limits, integration by substitution (see below) and squeeze theorem (Lemma 1).

Lemma 5 *Integral by Substitution*
$\vdash \forall$ f g f′ g′ a b. (a <= b)∧(∀ a b. a <= b ⇒ (g a <= g b))∧
(∀ x. g a <= x ∧ x <= g b ⇒ ((f diffl f′ x) x))∧
(∀ x. a <= x ∧ x <= b ⇒ ((g diffl g′ x) x))$\Longrightarrow$
(integral (g a,g b) f′ = integral (a,b) (λx.f′(g x)∗ g′ x))

The first two assumptions define limits of integration of the integrals on the right and the left hand side, respectively. Whereas, the remaining two assumptions ensure the differentiability of functions f and g in their respective allowable intervals. The formal proof of Lemma 5 involves the fundamental theorem of calculus, chain rule of differentiation and the uniqueness of the Gauge integral.

This completes our formalization of the improper integrals based on sequential limits. The overall proof script for the formalization reported in this section comprises of approximately 3500 lines of HOL code and took about 250 man-hours. Building on the top of our formalization of improper integrals and newly verified properties of Gauge integral, we present the higher-order logic formalization of the Gamma Function in the next section.

5 Formalization of the Gamma Function

The most commonly used definition of the Gamma function is given in (1). It is not possible to formalize this definition as it does not cover the case when the integrand ($t^{z-1}e^{-t}$) becomes unbounded at the lower limit of integration, i.e., when the argument of the Gamma function z is less than one. So, it is convenient to write (1) using the theory of improper integrals as follows:

$$\Gamma(z) = \lim_{a\to 0_+, b\to\infty} \int_a^b t^{z-1}e^{-t}dt \tag{12}$$

From the above definition, it is clear that the Gamma function involves both the cases of improper integrals discussed in Section 1, i.e., the interval is unbounded due to the upper limit of integration and integrand becomes unbounded at the lower limit of integration. Thus, we can use our formal definition of improper integrals (Definition 13) to formalize (12) as follows:

Definition 15 Gamma Function

$\vdash \forall$ z. gamma z =
    seq_improper_Integral $(\lambda n.\ \frac{1}{2^n}, \lambda m.\ m)(\lambda t. t$ rpow $(z-1) * \exp(-t))$

The function rpow is a power function with real exponent. It takes two real numbers x and y, and returns x^y. The HOL formalization of rpow is given as follows:

Definition 16 Real Powers

$\vdash \forall$ x y. x rpow y = if 0 < x then exp (y * ln x) else
    if x = 0 then if y = 0 then 1 else 0 else
    if $\exists$ m n. ODD m $\wedge$ ODD n $\wedge$ (abs y = $\frac{m}{n}$) then
    -(y * ln (-x)) else exp (y * ln (-x))

Here, the value of x^y is obvious for $0 < x$. The cases 0^0 and 0^y are defined as 0 and 1, respectively. This definition can also handle the case when the first argument is negative (e.g., $(-x)^2$).

The lower and upper incomplete Gamma functions play a vital role in obtaining fractional integration and differentiation of periodic functions, such as, sinusoidal response study of fractional operators [10]. They are mathematically defined as follows:

$$\gamma_l(x, z) = \int_{0^+}^{x} t^{z-1}e^{-t}dt \tag{13}$$

$$\Gamma_u(x, z) = \int_{x}^{\infty} t^{z-1}e^{-t}dt \tag{14}$$

and can be formalized in higher-order logic using our definition of improper integral as follows:

Definition 17 Lower Incomplete Gamma Function

$\vdash \forall$ x z. gamma_lower x z =
    seq_improper_Integral $(\lambda n.\ \frac{1}{2^n}, \lambda m.\ x)(\lambda t.\ t$ rpow$(z-1) * \exp(-t))$

Definition 18 Upper Incomplete Gamma Function
⊢ ∀ x z. gamma_upper x z =
    seq_improper_Integral (λn. x, λm. m) (λt. t rpow (z − 1) ∗ exp(−t))

Now we present the formal verification of some of the key properties of the Gamma function using the HOL4 theorem prover. The formal verification of these properties not only ensures the correctness of our formal definitions but also paves the way to the formal reasoning about physical systems involving the Gamma function in their analysis.

Theorem 6 *Pseudo-Recurrence Relation*
⊢ ∀ z . (0 < z) ⟹ (gamma (z+1) = z * gamma (z))

Proof Sketch We start the proof process by rewriting the left and right hand sides using Definition 13 and 15. We used INTEGRAL_BY_PARTS theorem, given in Table 1, along with some arithmetic reasoning to simplify the left-hand-side of the above goal as follows:

$$\begin{aligned}&\mathtt{lim}(\lambda n.\ \mathtt{lim}(\lambda m.\ (m+1)\ \mathtt{rpow}\ z * (-1 * \exp(-1 * (m+1)))\\ &\quad -(\frac{1}{2^n})\ \mathtt{rpow}\ z * (-1 * \exp(-1 * (\frac{1}{2^n})))\\ &+ x * \mathtt{integral}(\frac{1}{2^n}, m)(\lambda t.t\ \mathtt{rpow}\ (z-1) * \exp(-1 * t))))\end{aligned} \tag{15}$$

This step requires the proof of integrability of $\int_{\frac{1}{2^n}}^{m} t^{z-1}e^{-t}dt$ and $\int_{\frac{1}{2^n}}^{m} t^{z}e^{-t}dt$, which can be verified using the facts that integrand in each case is continuous on limits of integration and continuity implies the integrability. Now, we first verify the following two subgoals:

$$\lambda m.\ ((m+1)\ \mathtt{rpow}\ z) * (-1 * \exp(-1 * (m+1))) \longrightarrow 0 \tag{16}$$

$$\lambda n.\ ((\frac{1}{2^n})\ \mathtt{rpow}\ z) * (-1 * \exp(-1 * (\frac{1}{2^n}))) \longrightarrow 0 \tag{17}$$

based on the facts that both the terms in (16) approach to zero as m becomes very large and the first term in (17) approaches to zero while the second term in (17) approaches to 1 as n becomes very large. The availability of these subgoals allows us to verify that the first two terms of (15) approach 0 as n and m become very large. This way we are left with the following subgoal:

$$\begin{aligned}&\mathtt{lim}(\lambda n.\ \mathtt{lim}(\lambda m.\ z * \mathtt{integral}(\frac{1}{2^n}, m)(\lambda t.\ t\ \mathtt{rpow}\ (z-1) * \exp(-t)))) =\\ &z * \mathtt{lim}(\lambda n.\ \mathtt{lim}(\lambda m.\ \mathtt{integral}(\frac{1}{2^n}, m)(\lambda t.\ t\ \mathtt{rpow}\ (z-1) * \exp(-t))))\end{aligned} \tag{18}$$

The proof of this goal is a straightforward limit theory proof, given that we verify the existence of the limits, i.e.,

$$\forall z.(0 < z) \Rightarrow (\exists k.(\lambda m.\mathtt{integral}(\frac{1}{2^n}, m)(\lambda t.\ t\ \mathtt{rpow}\ (z-1) * \exp(-t))) \rightarrow k) \tag{19}$$

$$\forall z.(0 < z) \Rightarrow (\exists p.(\lambda n.\mathtt{integral}(\frac{1}{2^n}, m)(\lambda t.\ t\ \mathtt{rpow}\ (z-1) * \exp(-t))) \rightarrow p) \tag{20}$$

We split the main integral of above subgoals into sum of two integrals as follows:

$$\begin{aligned}\texttt{integral}(\frac{1}{2^{n}}, \texttt{m})(\lambda \texttt{t}.\ \texttt{t rpow}\ (\texttt{z}-1) * \exp(-\texttt{t})) = \\ \texttt{integral}(\frac{1}{2^{n}}, 1)(\lambda \texttt{t}.\ \texttt{t rpow}\ (\texttt{z}-1) * \exp(-\texttt{t})) + \\ \texttt{integral}(1, \texttt{m})(\lambda \texttt{t}.\ \texttt{t rpow}\ (\texttt{z}-1) * \exp(-\texttt{t}))\end{aligned} \tag{21}$$

In order to rewrite (19) and (20) with the above result, we need to verify the upper and lower bounds of each integral as follows:

$$\begin{aligned}0 \le \int_{\frac{1}{2^{n}}}^{1} \texttt{t rpow}\ (\texttt{z}-1) * e^{-t} dt \le \int_{\frac{1}{2^{n}}}^{1} \texttt{t rpow}\ (\texttt{z}-1) dt \\ 0 \le \int_{1}^{\infty} \texttt{t rpow}\ (\texttt{z}-1) * e^{-t} dt \le \int_{1}^{\infty} \frac{\texttt{FACT(n)}}{\texttt{t rpow n} + 1 - \texttt{z}} dt\end{aligned} \tag{22}$$

The verification of above bounds requires an interesting lemma, i.e., $[\forall t\ n. 0 < t \Rightarrow \frac{t^{n}}{FACT(n)} < exp\ t]$, which we verified using the definition of exp and properties of summations. This concludes the verification of Theorem 3, which was primarily based on extensive arithmetic rewriting along with formal reasoning related to integrals, derivatives and limits. □

Theorem 7 *Functional Equation*
⊢ gamma 1 = 1

We start the proof process by rewriting the left-hand-side using the definition of the Gamma function and improper integrals (Definition 13 and 15).

$$\texttt{lim}\ (\lambda \texttt{n}.\ \texttt{lim}\ (\lambda \texttt{m}.\texttt{integral}(\frac{1}{2^{n}}, \texttt{m})(\lambda \texttt{t}.\texttt{t rpow}\ (\texttt{z}-1) * \exp(-\texttt{t})))) = 1 \tag{23}$$

Using Modus Ponens rule with the Fundamental Theorem of Calculus (FTC1) and properties of differentiation, we can simplify the the proof goal of (23) as follows:

$$\texttt{lim}\ (\lambda \texttt{n}.\ \texttt{lim}\ (\lambda \texttt{m}.\ -\exp\ (-(\texttt{m}+1))\ +\ \exp\ (-(\frac{1}{2^{n}}))\ =\ 1 \tag{24}$$

The above goal is a straightforward limit theory proof, which can be verified based on the linearity properties of limits and some arithmetic reasoning.

Theorem 8 *Generalization of Factorial*
⊢ ∀ k ∈N . gamma (k + 1) = FACT(k)

The proof of Theorem 8 involves induction on the variable k. The base case can be discharged by rewriting with the definition of Factorial and Theorem 7. In the step case, we need to prove the following subgoal:

$$\texttt{gamma}\ (\texttt{k}\ +\ 1\ +\ 1)\ =\ \texttt{FACT}(\texttt{k}\ +\ 1)$$

This can be simplified to $(\texttt{k}+1) * \texttt{gamma}\ (\texttt{k}+1) = (\texttt{k}+1) * (\texttt{FACT k})$ using Theorem 6, which in turn can be verified based on simple arithmetic reasoning.

Theorem 9 *Reconstruction of the Gamma Function*
⊢ ∀ a z.(0 < z) ∧ (0 < a) ⟹
    (gamma z = gamma_upper a z + gamma_lower a z)

We start the verification by rewriting the left-hand-side with Definitions 13 and 15.

$$\begin{aligned}\mathtt{lim}(\lambda \mathtt{m}.\ \mathtt{lim}(\lambda \mathtt{m}.\ \mathtt{integral}(\frac{1}{2^{\mathtt{n}}}, \mathtt{m})(\lambda \mathtt{t}.\ \mathtt{t}\ \mathtt{rpow}\ (\mathtt{z}-1) * \mathtt{exp}(-1 * \mathtt{t})))) \\ = \mathtt{gamma\_upper}\ \mathtt{a}\ \mathtt{z} + \mathtt{gamma\_lower}\ \mathtt{a}\ \mathtt{z}\end{aligned} \tag{25}$$

Now the main step is to split the integral on the left-hand-side into the sum of two integrals as follows:

$$\begin{aligned}\mathtt{integral}(\frac{1}{2^{\mathtt{n}}}, \mathtt{m})(\lambda \mathtt{t}.\ \mathtt{t}\ \mathtt{rpow}\ (\mathtt{z}-1) * \mathtt{exp}(-1 * \mathtt{t})) = \\ \mathtt{integral}(\frac{1}{2^{\mathtt{n}}}, \mathtt{a})(\lambda \mathtt{t}.\ \mathtt{t}\ \mathtt{rpow}\ (\mathtt{z}-1) * \mathtt{exp}(-1 * \mathtt{t})) + \\ \mathtt{integral}(\mathtt{a}, \mathtt{m})(\lambda \mathtt{t}.\ \mathtt{t}\ \mathtt{rpow}\ (\mathtt{z}-1) * \mathtt{exp}(-1 * \mathtt{t}))\end{aligned} \tag{26}$$

This subgoal can be verified using the `INTEGRAL_COMBINE` property of the Gauge integral (Table 1). But we need to prove that $\frac{1}{2^{\mathtt{n}}} \leq \mathtt{a}$, for which we verify the property of sequence $[\forall \mathtt{m}.(\lambda \mathtt{n}.\mathtt{f}\ \mathtt{n}) \to \mathtt{p} \Leftrightarrow (\lambda \mathtt{n}.\mathtt{f}(\mathtt{n}+\mathtt{m})) \to \mathtt{p}]$ and then by rewriting the left-hand-side of (25), we need to verify the goal $[(0 < \mathtt{a}) \Rightarrow (1/2^{\mathtt{n}+\mathtt{clg}(\mathtt{ln}\ \mathtt{a}/\mathtt{ln}(1/2))} \leq \mathtt{a})]$, which can be proved using the properties of logarithm and simple arithmetic reasoning. After simplifying the subgoal of (25) using (26), we require the linearity properties of limits along with the definitions of incomplete Gamma functions (Definition 17 and 18) to complete the proof of Theorem 9.

Theorem 10 *Recurrence Relation of Lower Incomplete Gamma Function*

```
⊢ ∀ z x. 0 < z ∧ 0 < x ⟹ gamma_lower x (z+1) =
     z * gamma_lower x z - x rpow (z) * exp(-x))
```

Theorem 11 *Recurrence Relation of Upper Incomplete Gamma Function*

```
(⊢ ∀ s x. 0 < z ∧ 0 < s ⟹ (gamma_upper s z =
     (z-1) * gamma_upper s (z-1) + s rpow (z-1) * exp(-s))
```

The verification steps for Theorems 10 and 11 are very similar to the ones for Theorem 6. The major steps are to simplify the integral on the left-hand-sides using `integral_by_parts` (Similar to (15)) and the verification of the convergence of integrals.

This completes our formalization of the Gamma function, which to the best of our knowledge is the first one in higher-order logic. The Gamma function is useful in many domains, such as, probability theory (Gamma Distribution) and fractional calculus [9], and our formalization can be directly utilized in such applications. Our formalization of the Gamma function can also be generalized to formalize other higher transcendental functions, such as, the Beta function.

Due to inherent soundness of higher-order logic theorem proving, our verification results are exactly the same as produced by the paper-and-pencil proof methods [3, 10]. The formalization, presented in this section, took around 5500 lines of HOL code and approximately 300 man-hours. However, the main advantage of this rigorous exercise is that our results can be built upon to facilitate formal reasoning about applications involving the Gamma function. Our proof script is available for download [44] and thus can be utilized by other researchers working in this field.

6 Applications

We illustrate the usefulness of our formalization of the Gamma function by applying it to formally reason about three commonly used mathematical results, i.e., Euler's Generalized rule of differentiation, Mittag-Leffler functions and the relationship between the Exponential and Gamma random variables.

6.1 Euler's Generalized Rule of Differentiation

We are familiar with the meaning and notion of derivatives, i.e., $\frac{df(x)}{dx}$, $Df(x)$, $\frac{d^2f(x)}{dx^2}$ and $D^2f(x)$. But we rarely think that what could be the meaning of derivatives of non-integer order, i.e., $\frac{d^{\frac{1}{2}}f(x)}{dx^{\frac{1}{2}}}$ or $\frac{d^{\pi}f(x)}{dx^{\pi}}$. It looks a new idea but yet the history of non-integer order derivatives is as old as the integer-order Calculus. In 1695, Leibnitz, who is known as the founder of Calculus, discussed his idea of denoting n^{th} order derivative as $\frac{d^nf(x)}{dx^n}$ with L'Hôpital, who, in return asked Leibnitz, what if n is $\frac{1}{2}$. Later on, Leibnitz continued his work on the idea of representing derivatives of non-integer order and his work gave birth to an interesting area of mathematics, known as fractional calculus, which deals with the derivatives and integrals of non-integer order including rational, irrational and even complex numbers.

The concept of fractional calculus has great potential to change the way we see, model and analyze the systems. We can say that ignoring fractional calculus is just like ignoring fractional, irrational or complex numbers. It provides a good opportunity to scientists and engineers for revisiting the origins. The theoretical and practical interests of using fractional order operators are increasing. The application domain of fractional calculus is ranging from accurate modeling of the microbiological processes [33] to the analysis of astronomical images [48]. Different formalisms have been proposed for computing non integer order derivatives and integrals, namely, Riemann-Liouville and Grünwald-Letnikov [10]. One of the earliest definition of fractional derivative of a function, that can be expressed as ($\mathtt{f(x) = ax^m}$), was proposed by Euler [9] in 1730, as follows:

$$\frac{d^v x^m}{dx^v} = \frac{\Gamma(m+1)}{\Gamma(m-v+1)} x^{m-v} \qquad v \in \mathbb{R}+ \tag{27}$$

Euler's definition is widely used for calculating fractional derivatives because of its simplicity compared to the other definitions of fractional derivatives. For example, it can be directly utilized to reason about the fractional derivative of parabolic step-type transition (used in edge detection [35]).

Our formalization of the Gamma function, given in Section 5, allows us to formalize Euler's derivative as follows:

Definition 19 Euler's Derivative
$\vdash \forall$ v x m. euler_deriv v x m = $\frac{\text{gamma (m+1)}}{\text{gamma (m-v+1)}}$ * (x rpow (m-v))

Based on the above formalization, we now verify a couple of classical Euler's derivative properties which prove that Euler's derivative is consistent with classical integer order derivative and hence facilitate the usage of Definition 19 to real-world problems.

Theorem 12 *Relation of Euler's Derivative with Integer Order Derivative*

```
⊢ ∀ v ∈ℕ x m. (v ≤ m) ⟹
    euler_deriv v x m = FACT(m)/FACT(m-v) * (x rpow (m-v))
```

Theorem 13 *Identity of Euler's Derivative*

```
⊢ ∀ x m. euler_deriv 0 x m = x rpow m
```

We start the verification of Theorem 11 by rewriting the left-hand-side with Definition 19 (Euler's Derivative) which results in the following subgoal:

$$\frac{\mathtt{gamma(m+1)}}{\mathtt{gamma(m-v+1)}} * \mathtt{x\,rpow\,(m-v)} = \frac{\mathtt{FACT(m)}}{\mathtt{FACT(m\text{-}v)}} * \mathtt{(x\,rpow\,(m-v))} \tag{28}$$

The above subgoal can now be verified using Theorem 6. The verification of Theorem 12 utilizes Theorems 6, 7 and 8, verified in the last section. Overall, the verification of these theorems took a few lines of HOL code, which demonstrates the effectiveness of our formalization of the Gamma function.

6.2 Mittag-Leffler Functions

The Mittag-Leffler functions, that are usually denoted by $E_\alpha(z)$, $E_{\alpha,\beta}(z)$ are named in honor of Gösta Mittag-Leffler, the well-known Swedish mathematician. The first function $E_\alpha(z)$, also named as, one-parameter Mittag-Leffler function, was introduced by Mittag-Leffler [40] in 1903, in his work related to summation of divergent series. Whereas, the two parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$, was first discussed in the work of Wiman [51]. The one-parameter and two-parameter Mittag-Leffler functions are defined as follows:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \tag{29}$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{30}$$

During the last two decades, the theoretical and practical interest of using Mittag-Leffler function is significantly increased among scientists and engineers due to huge potential applications in several real-world problems, such as fractional order integral and differential equations, fractional generalization of the kinetic equation and super-diffusive transport. The one-parameter and two-parameter Mittag-Leffler functions relate a purely exponential law and power-law like behavior of a given phenomena which are governed by ordinary and fractional kinetic equations [25]. Now, we present a higher-order-logic formalization of Mittag-Leffler functions and we also formally verify some of their important properties, which shows the usefulness of formalized definition of the Gamma function and its corresponding properties.

Definition 20 One-Parameter Mittag-Leffler function

```
⊢ ∀ z alpha. mittag_leffler_1 z alpha =
     suminf (λk. inv (gamma (alpha * &k + 1)) * z^k)
```

Definition 21 Two-Parameter Mittag-Leffler function

```
⊢ ∀ z alpha beta. mittag_leffler_2 z alpha beta =
     suminf (λk. inv (gamma (alpha * &k + beta)) * z^k)
```

Where `suminf` takes a function `f` of type $\mathbb{N} \rightarrow \mathbb{R}$ and returns the infinite sum f(0)+f(1)+f(2)+ . . . . , assuming it converges. The function `inv` reciprocates a real number and, `&` is the type casting from $\mathbb{N}$ to $\mathbb{R}$.

Mittag-Leffler functions are famous for their characteristic of generalizing the Exponential function (`exp`). That is why, Mittag-Leffler functions play the same role in fractional calculus as the Exponential function (`exp`) does in integer order calculus. The formal verification of this property of Mittag-Leffler functions is presented in the following two theorems.

Theorem 14 *Mittag-Leffler_1_exp*
```
⊢ ∀ z. mittag_leffler_1 z 1 = exp z
```

Theorem 15 *Mittag-Leffler_2_exp*
```
⊢ ∀ z. mittag_leffler_1 z 1 1 = exp z
```

The verification of Theorems 14 and 15 requires a few lines of HOL code and mainly involves rewriting with the definition of Mittag-Leffler functions, i.e., Definitions 20 and 21, along with the factorial generalization (Theorem 8) property of the Gamma function.

Another important property of the two parameter Mittaf-Leffler function is the generalization of hyperbolic functions, i.e., $cosh(z)$ and $sinh(z)$. Next,we verify this property for the case of $cosh$.

Definition 22 Cos_Hyperbolic_Function
```
⊢ ∀ z. cosh z = suminf (λk. inv (& FACT(2*k))* z^(2*k))
```

Theorem 16 Mittag-Leffler_2_cosh
```
⊢ ∀ z. mittag_leffler_1 z^2 2 1 = cosh z
```

Again, the availability of the Gamma function greatly facilitated the formal reasoning process, and the formal verification of Theorem 16 requires only rewriting with the formal definition of $E_{\alpha,\beta}(z)$, $cosh(z)$ and Theorem 8 (factorial generalization).

6.3 Relationship Between Exponential and Gamma Random Variables

Formal probabilistic analysis using theorem proving [21] is an emerging direction in the area of formal methods. The main idea is to use formalized random variables to model uncertainties and random components of a given system and then formally verify its probabilistic and statistical properties associated with the desired characteristics of the system. The foremost criteria for conducting probabilistic analysis within a higher-order-logic theorem prover is to be able to express probabilistic notions, such as probability of an event and random variables, in higher-order logic and reason about the probability distribution and statistical properties of random variables in a higher-order-logic theorem prover. A formalized probability theory provides the foundations for expressing probabilistic notions. A number of authors, including Hurd [27], Mhamdi [38] and Hölze [26], reported higher-order-logic based formalizations of probability theory. The recent works by Mhamdi [38] and Hölzl [26] are based on extended real numbers (including $\pm\infty$) and provide the formalization of Lebesgue integral for reasoning about advanced statistical properties. This way, they are more mature than Hurd's [27] formalization of measure and probability theories, which is based on simple real numbers. However, these recent formalizations do not support a particular probability space like the one presented in Hurd's work. Due to this distinguishing feature, Hurd's formalization [27] has been utilized to verify sampling algorithms of a number of commonly used discrete [27] and continuous random variables [23] based on their probabilistic and statistical properties [22, 24]. However, to the best of our knowledge, due the unavailability of a higher-order-logic formalization of the Gamma function the Gamma random variable has not been formalized in higher-order-logic so far. In this section, we utilize our Gamma function formalization to formalize the relationship between Exponential and Gamma random variables based on Hurd's formalization of probability theory [27].

The Gamma(k, θ) random variable (X) is a continuous random variable with the following Cumulative Distribution Function (CDF)

$$Pr(X \leq x) = \frac{\gamma_l(k, \frac{x}{\theta})}{\Gamma k} \tag{31}$$

where γ_l and Γ denote the lower incomplete Gamma function and Gamma function, respectively. The Gamma random variable is widely used in many scientific and engineering domains including wireless networks [2], neuroscience [29] and genetic engineering [14]. The Exponential random variable represents a special case of the general Gamma(k, θ) random variable with the shape parameter $k = 1$ and we present the formal verification of this result here. For this purpose, we first define the Gamma$(1, \theta)$ random variable in terms of exponential random variable as follows:

Definition 23 Gamma Random Variable
⊢ ∀ t s. gamma_rv t s = exp_rv ($\frac{1}{t}$) s

where the function `exp_rv l s` represents the Exponential random variable with parameter l sampled from the infinite Boolean sequence s [23]. In order to formally verify the above mentioned relationship, we formally verified the CDF relation for the function `gamma_rv` as the following theorem:

Theorem 17 *CDF of the Gamma Random Variable*
⊢ ∀ x t. (0 < t) ⇒
  (prob bern {s | gamma_rv t s ≤ x} =
   (if x ≤ 0 then 0 else ((gamma_lower ($\frac{x}{t}$) 1)/(gamma 1))))

where `prob bern` represents the probability function in the infinite Boolean sequence space `bern`. The above theorem indicates that the CDF of the Gamma random variable specified in Definition 23 is a special case of the general CDF of the Gamma random variable, given in (31), when $k = 1$ and thus confirms the relationship between the two random variables. The verification of Theorem 17 is is primarily based on Theorems 2 and 6 along with some arithmetic and probability theoretic reasoning. The proof script is very straightforward and is provided in [44], which clearly indicates the usefulness of our Gamma function formalization. Theorem 16 along with the additive property of the Gamma random variable can be used to verify the CDF of the general Gamma(k, θ) random variable by modeling it as a sum of k Exponential random variables. Currently, the probabilistic analysis using theorem proving framework [21] does not support handling multiple continuous random variables and we are working on this extension, which once done could be used with the results of the current paper to formalize the general Gamma random variable.

In this section, so far, we presented the applications of our formalization of improper integrals and Gamma function. It is worth mentioning that the formalization of the sequential improper integrals and the Gamma function took thousands of lines of HOL code but this effort allowed us to formally verify interesting properties of Euler's generalized rule of differentiation, Mittag-Leffler function and the relationship between the Exponential and Gamma random variables in a very straightforward way requiring only 500 lines of HOL code. All the applications presented in this section are heavily used in the domains of fractional calculus and probabilistic analysis and the contributions of this section can be used to extend the recently reported formal frameworks for the analysis of fractional order systems [45] and formal probabilistic analysis [21]. Besides the importance in formal analysis of fractional calculus based systems and formal probabilistic analysis, the current work also highlights the strength and sophistication of present age proof assistants. The interest of using proof assistants for the formalization of classical theories of mathematics is a natural corollary as the formal verification of safety-critical physical systems generally relies upon such theories. Another motivation behind the formalized mathematical theories is the soundness of theorem proving systems and ability to use these theories without discrepancies which are usually encountered in our usual mathematical analysis.

7 Conclusions

In this paper, we extended the existing higher-order-logic formalization of the Gauge integral by formalizing a variant of improper integrals using the sequential limits and formally verified some of the classical properties such as improper integral theorem and improper shift theorem. Consequently, this extension in turn allowed us to formalize the Gamma function and formally verify some of its key properties using the HOL4 theorem prover, which is a novelty to the best of our knowledge. For illustration purposes, we formalize the Euler's derivative, Mittag-Leffler functions and the relationship between the Exponential and Gamma random variables. The reported formalization also includes the formalization of some of the most interesting historical mathematical results, e.g., formal proof of the factorial generalization of the Gamma function and formalization of Mittag-Leffler function, which is a generalization of the Exponential function.

The reported formalization opens the doors to many interesting and novel directions of research. Some worth mentioning ones include enriching the library of the formally verified properties of the Gamma function and incomplete Gamma functions with the relationship to Beta function and incomplete Beta functions. Similarly, the formally verified relationship between the Exponential and the Gamma random variables can be utilized to formalize the general Gamma(k, θ) distribution, which is the sum of k Exponential random variables. Our formalization was done using real numbers and the same formalization can also be extended to cover the complex numbers using the higher-order-logic formalization of complex number theory [19]. The extension of the current work to complex domain is very challenging in HOL4 theorem prover due to unavailability of the formalization of multivariate analysis libraries. On the other hand, HOL-Light provides rich support of multivariate libraries which are required for the complex extension of the current work. However, formalization of the Gamma function in HOL-Light requires expertise in both multivariate analysis and higher-order logic theorem proving. One of our main motivations for formalizing the Gamma function is to be able to formalize the commonly used Gamma random variable, which has applications in many safety-critical areas including wireless sensor networks, neuroscience and genetic engineering. Therefore, the compatibility of the Gamma function with the formalized probability theory was very important. Our prior work on probabilistic analysis primarily builds upon Hurd's formalization of probability theory [27], which is in turn based on real numbers and is available in the HOL4 theorem prover. There have been many developments in the probabilistic analysis using theorem proving so far including the formalization of continuous random variables [21, 22], Markov chains [30] and some aspects of wireless sensor networks [11]. The formalization of the Gamma random variable fits very well with these available theories and therefore we chose the HOL4 theorem prover for the formalization of the Gamma random variable and ported the required foundations from the HOL-Light theorem prover. Our extensions to the Gauge integral formalization in HOL4 can also be built upon to achieve many interesting directions in the area of formally reasoning about continuous physical aspects of real-world systems.

References

1. Uncertain Singular Expressions in Mathematica. (2014). http://reference.wolfram.com/mathematica/ref/FullSimplify.html
2. Al-Ahmadi, S., Yanikomeroglu, H.: On the approximation of the generalized-k distribution by a gamma distribution for modeling composite fading channels. Trans. Wirel. Commun. **9**(2), 706–713 (2010)
3. Artin, E.: The Gamma Function. Athena Series (1964)
4. Baumann, G.: Fractional calculus and symbolic solution of fractional differential equations. In: Fractals in Biology and Medicine, Mathematics and Biosciences in Interaction, pp. 287–298. Birkhäuser Basel (2005)
5. Butler, R.W.: Formalization of the integral calculus in the PVS theorem prover. J. Formalized Reason. **2**(1), 1–26 (2009)
6. Church, A.: A formulation of the simple theory of types. J. Symb. Log. **5**, 56–68 (1940)
7. Cruz-Filipe, L.: Constructive Real Analysis: A Type-Theoretical Formalization and Applications. Ph.D. Thesis, University of Nijmegen (2004)
8. Cheng, Q., Cui, T.J., Zhang, C.: Waves in planar waveguide containing Chiral Nihility metamaterial. Opt. Commun. **276**(2), 317–321 (2007)
9. Dalir, M., Bashour, M.: Application of fractional calculus. Appl. Fractional Calc. Phys. **4**(21), 12 (2010)
10. Das, S.: Functional Fractional Calculus for System Identification and Controls, 1st edn (2007)
11. Elleuch, M., Hasan, O., Tahar, S., Abid, M.: Formal analysis of a scheduling algorithm for wireless sensor networks. In: Formal Engineering Methods LNCS, vol. 6991, pp. 388–403. Springer (2011)
12. Engheta, N.: On the Role of Fractional Calculus in Electromagnetic Theory. Antennas and Propagation Magazine. IEEE
13. Gordon, M.J.C., Melham, T.F. (eds.): Introduction to HOL A Theorem Proving Environment for Higher-Order Logic. Cambridge University Press (1993)
14. Hallen, M., Li, B., Tanouchi, Y., Tan, C., West, M., You, L.: Computation of steady-state probability distributions in stochastic models of cellular networks. PLoS Comput. Biol. **7**(10), e1002,209, 1–16 (2011)
15. Harrison, J.: Formalized Mathematics. Tech. Rep. 36, Turku Centre for Computer Science (1996)
16. Harrison, J.: HOL light: A tutorial introduction. In: Srivas, M., Camilleri, A. (eds.), Proceedings of the First International Conference on Formal Methods in Computer-Aided Design (FMCAD'96), Lecture Notes in Computer Science, vol. 1166, pp. 265–269. Springer-Verlag (1996)
17. Harrison, J.: Theorem Proving with the Real Numbers. Springer-Verlag (1998)
18. Harrison, J.: Formalizing basic complex analysis. In: From Insight to Proof: Festschrift in Honour of Andrzej Trybulec, Studies in Logic, Grammar and Rhetoric, vol. 10, no. 23, pp. 151–165. University of Białystok (2007)
19. Harrison, J.: The HOL light theory of Euclidean space. J. Autom. Reason. **50**(2), 173–190 (2013)
20. Hasan, O.: Formal Probabilistic Analysis using Theorem Proving. Ph.D. Thesis, Concordia University, Montreal, P.Q., Canada (2008)
21. Hasan, O.: Formal Probabilistic Analysis using Theorem Proving. Ph.D. Thesis, Concordia University, Montreal, QC, Canada (2008)

22. Hasan, O., Abbasi, N., Akbarpour, B., Tahar, S., Akbarpour, R.: Formal reasoning about expectation properties for continuous random variables. In: Formal Methods, LNCS, vol. 5850, pp. 435–450. Springer (2009)
23. Hasan, O., Tahar, S.: Formalization of the continuous probability distributions. In: Automated Deduction, LNAI, vol. 4603, pp. 3–18. Springer (2007)
24. Hasan, O., Tahar, S.: Using theorem proving to verify expectation and variance for discrete random variables. J. Autom. Reason. **41**(3–4), 295–323 (2008)
25. Haubold, H.J., A.M. Saxena, R.: Mittag-leffler functions and their applications. J. Appl. Math. **2011**, 51 (2011)
26. Holzl, J., Heller, A.: Three chapters of measure theory in Isabelle/HOL. In: Interactive Theorem Proving, LNCS, vol. 6172, pp. 135–151. Springer (2011)
27. Hurd, J.: Formal Verification of Probabilistic Algorithms. Ph.D. Thesis, University of Cambridge, Cambridge, UK (2002)
28. Jr., E.M.C., Grumberg, O., Peled, D.A.: Model Checking. The MIT Press (1999)
29. Levine, M.W.: Variability in the maintained discharges of retinal ganglion cells. J. Opt. Soc. Am. A **4**(12), 2308–2320 (1987)
30. Liu, L., Hasan, O., Tahar, S.: Formalization of finite-state discrete-time Markov chains in HOL. In: Automated Technology for Verification and Analysis, LNCS, vol. 6996, pp. 90–104. Springer (2011)
31. Lozier, D.W., Olver, F.W.J.: Numerical evaluation of special functions., Gautschi, W. (ed.) AMS Proceedings of Symposia in Applied Mathematics, vol. 48. pp. 79–125, American (1994)
32. Chaudry, M.A., S.M.Z.: On A Class of Incomplete Gamma Functions with Applications. CRC Press (2001)
33. Magin, R.L.: Fractional calculus models of complex dynamics in biological tissues. Comput. Math. Appl. **59**, 1586–1593 (2010)
34. Magin, R.L., Ovadia, M.: Modeling the cardiac tissue electrode interface using fractional calculus. J. Vib. Control. **14**(9-10), 1431–1442 (2008)
35. Mathieu, B., Melchior, P., Oustaloup, A., Ceyral, C.: Fractional differentiation for edge detection. Signal Process. **83**(11), 2421–2432 (2003)
36. MATLAB (2014). http://www.mathworks.com/products/matlab/
37. Mhamdi, T., Hasan, O., Tahar, S.: On the formalization of the Lebesgue integration theory in HOL. In: Interactive Theorem Proving, pp. 387–402. Springer, LNCS (2010)
38. Mhamdi, T., Hasan, O., Tahar, S.: On the formalization of the Lebesgue integration theory in HOL. In: Interactive Theorem Proving, LNCS, vol. 6172, pp. 387–402. Springer (2011)
39. Milner, R.: A theory of type polymorphism in programming. J. Comput. Syst. Sci. **17**, 348–375 (1978)
40. Mittag-Leffler, G.: Sur La Nouvelle Fonction $E_{\alpha(x)}$. C.R. Acad. Sci. Paris **137**, 554–558 (1903)
41. Naqvi, A.: Comments on waves in planar waveguide containing Chiral Nihility metamaterial. Opt. Commun. **284**, 215–216 (2011)
42. Paulson, L.C.: ML for the Working Programmer. Cambridge University Press (1996)
43. Shintani, T.: On Kronecker limit formula for real quadratic fields (1976)
44. Siddique U.: Formalization of Gamma function in Higher-order Logic - HOL Proof Script (2014). http://save.seecs.nust.edu.pk/students/umair/gamma.html
45. Siddique, U., Hasan, O.: Formal analysis of fractional order systems in HOL. In: Formal Methods in Computer Aided Design (FMCAD), pp. 163–170 (2011)
46. Simpson, C.: Computer theorem proving in mathematics. Lett. Math. Phys. **69**(1), 287–315 (2004). doi:10.1007/s11005-004-0607-9
47. Slind, K., Norrish, M.: A brief overview of HOL4. In: TPHOLs, pp. 28–32 (2008)
48. Sparavigna, A.C.: Fractional Differentiation Based Image Processing. arXiv:0910.2381 (2009)
49. Van Der Laan, C.G., Temme, N.M.: Calculation of Special Functions: The Gamma Function, the Exponential Integrals and Error-like Functions. Centrum voor Wiskunde en Informatica, Amsterdam. The Netherlands (1984)
50. Walck, C.: Hand-Book on Statistical Distributions for Experimentalists (1996)
51. Wiman, A.: Über Den Fundamental Satz in Der Theorie Der Funcktionen $E_{\alpha(x)}$. Acta Mathematica **29**, 191–201 (1905)