Formal Kinematic Analysis of the Two-Link Planar Manipulator

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Abstract. Kinematic analysis is used for trajectory planning of robotic manipulators and is an integral step of their design. The main idea behind kinematic analysis is to study the motion of the robot based on the geometrical relationship of the robotic links and their joints. Given the continuous nature of kinematic analysis, traditional computer-based verification methods, such as simulation, numerical methods or model checking, fail to provide reliable results. This fact makes robotic designs error prone, which may lead to disastrous consequences given the safety-critical nature of robotic applications. Leveraging upon the high expressiveness of higher-order logic, we propose to use higher-order-logic theorem proving for conducting formal kinematic analysis. As a first step towards this direction, we utilize the geometry theory of HOL-Light to develop formal reasoning support for the kinematic analysis of a two-link planar manipulator, which forms the basis for many mechanical structures in robotics. To illustrate the usefulness of our foundational formalization, we present the formal kinematic analysis of a biped walking robot.

1 Introduction

Kinematic analysis [14] is the study of motion of a machine or mechanism without considering the forces that cause the motion. It mainly allows us to determine parameters like the position, displacement, rotation, speed, velocity and acceleration of a given mechanical structure and is thus used to design the geometrical dimensions and operational range of a mechanical structure according to the given specifications. The main idea behind kinematic analysis is to first identify the links (rigid bodies) and joints (allow rotation or sliding) of the given mechanical structure and then construct a corresponding kinematic (skeleton) diagram, which is a geometrical structure depicting the connectivity of links and joints. Finally, the kinematic diagrams are analyzed, using the principles of geometry, to determine the motion of any point of interest in the kinematic diagram.

Kinematic analysis allows us to extract useful information about the workspace, dexterity and precision of a given robotic design [26]. Thus, kinematic analysis is always performed during the conception phase of a robot to ascertain that the designed robot is appropriate to serve the given purpose [22]. For example, kinematic analysis has been used to judge the slope climbing capability of a biped robot [19] and the repairing the human aortic aneurysm capability of a minimal invasive surgical robot [7].

Given the safety-critical nature of many robotic applications, traditional techniques, like numerical methods or simulations, are not encouraged to be used for kinematic analysis [24]. Computer algebra systems, like Maple and Mathematica, offer complete packages (e.g. [25]) for kinematic analysis of mechanical systems. Despite being very efficient for computing mathematical solutions symbolically, these methods cannot be considered 100% reliable due to the involvement of unverified huge symbolic manipulation algorithms in their core. As an alternative, interval analysis has been used to find the *safe* kinematics for a minimum invasive surgical robot [11]. However sometimes, due to computational pessimism, the resulting interval becomes too large to provide any useful information. Moreover, interval analysis is not suitable to exhaustively check the initial hypothesis of the model properly and thus cannot be completely relied upon as well. Inaccuracies in kinematic analysis could lead to disastrous consequences, including a robot's breakdown [13], and thus investigating more reliable and sound kinematic analysis techniques is a dire need.

In the past couple of decades, formal methods have emerged as a successful verification technique for both hardware and software systems. The rigorous exercise of developing a mathematical model for the given system and analyzing this model using mathematical reasoning usually increases the chances for catching subtle but critical design errors that are often ignored by traditional techniques like paper-and-pencil based proofs or numerical methods. However, due to the continuous nature of the analysis and the involvement of analytical geometry, automatic state-based formal methods, like model checking, cannot be used to ascertain absolute correctness. On the other hand, leveraging upon the high expressiveness of higher-order logic, theorem proving can provide the ability to formally reason about the correctness of kinematic analysis. But to the best of our knowledge, the underlying principles of kinematic analysis have not been formalized in higher-order-logic so far and thus formal reasoning about the correctness of kinematic analysis is not a straightforward task.

As a first step towards using a higher-order-logic theorem prover for formally verifying the correctness of kinematic analysis, we present the formal reasoning support for a two-link planar manipulator [22], i.e., a simple yet the most commonly used mechanical structure in robotics. In particular, we present the formalization of forward and inverse kinematic analysis equations of a two-link planar manipulator by extending the recently developed analytical geometry theories available in HOL-Light [9]. The main advantage of these results is that they greatly minimize the user intervention for formal reasoning about kinematic analysis of many robots, mainly because any robot with multiple links and joints can be expressed in terms of a two-link planar manipulator. In order to demonstrate the practical effectiveness and utilization of the reported formalization, we utilize it to conduct the formal kinematic analysis of a biped robot [12, 10], i.e., a two-legged mobile robot, in this paper.

The rest of the paper is organized as follows: Some related work about formal verification of mechanical systems and formalization of geometry theory is presented in Section 2. In Section 3, we provide a brief introduction about kinematic analysis of a two-link planar manipulator. The formalization of these foundations of kinematic analysis is provided in Section 4. We utilize this formalization to conduct the formal kinematic analysis of a biped robot in Section 5. Finally, Section 6 concludes the paper.

2 Related Work

The usage of formal methods in ascertaining the correctness of continuous and physical systems is increasingly being advocated these days [2]. In this context, formal verification of mechanical systems, particularly the ones used in automotive and robotic applications, have gained particular interest due to their safety-critical applications [8]. For example, formal verification of the movements of a Samsung Home-service Robot (SHR) is presented by analyzing its discrete control software using the Esterel model checker [6]. Similarly, an abstracted integer-valued behavior of the mobile outdoor robot RAVON is formally modeled in the synchronous language Quartz and is formally verified using the Averest model checker [17]. Moreover, in order to alleviate the problems associated with unintended acceleration due to faulty accelerator pedals, the electrical and mechanical components of Toyota's electronic throttle controller (ETC) have been formally modeled and verified based on the principles of timed automata and real-time logic [18]. Likewise, an abstraction approach for generating a discretized state-space of mechanical systems is reported in [21]. In all these model checking based verification efforts, the continuous dynamics of mechanical systems had to be discretized in order to be able to construct a corresponding automata-based model [20]. Such abstractions clearly compromise the accuracy of the analysis. These limitations can be overcome by using higher-order-logic theorem proving in the context of verifying mechanical systems. For example, the Isabelle theorem prover has been used to formally verify a collision-avoidance algorithm for service robots [23]. Real number and set theories have been utilized to formalize the contour of the robot as a convex polygon while obstacles are modeled as connected sets of points. This way, it has been formally verified that the moving robot is able to stop, upon detecting an obstacle, within the safety zone. The results have been verified without using any abstractions, which clearly indicates the usefulness of theorem proving in the context of verifying mechanical systems. With the same motivation, we plan to utilize higher-order-logic theorem proving for kinematic analysis in this paper, which, to the best of our knowledge, is a novelty.

The foremost requirement for conducting kinematic analysis in a higher-order-logic theorem prover is the ability to formally reason about geometry theory principles in a theorem prover. This capability is provided by a number of theorem provers. For example, a formal proof environment for Euclid's elements is presented in [1]. Some other recent developments include the axiomatic formalization of Euclidian plane geometry along with some interactive and automated reasoning support for geometrical properties in the Coq theorem prover[16], the formalization of the Cartesian-plane based geometry theory along with the proof that it can model the synthetic plane geometries in the Isabelle/HOL theorem prover [15], and the HOL-Light geometry theory formalized, based on n-dimensional real vector ($\texttt{real}^{\texttt{n}}$), in the Euclidean space [9]. In this paper, we have chosen the HOL-Light theorem prover for developing the foundations of kinematic analysis. The main reasons behind this choice include the availability of all the topological and analytic foundations for vectors, which is expected to play a vital role in extending the reported formalization for analyzing other continuous aspects of mechanical systems, and our past familiarity with the HOL-Light.

3 Kinematic Analysis of Two-Link Planar Manipulator

A two-link manipulator [22], depicted in Figure 1, has two rotary degrees of freedom (DOF) in the same plane. In kinematic analysis of this system, we are mainly interested in the trajectory planning of the end-effector (tip) of the manipulator. In order to bring the end-effector from its initial Cartesian position to an arbitrary point in the Cartesian space, we need to find its relationships with the joint angles, i.e., θ_1 and θ_2. This can be done in two ways, i.e. forward or inverse kinematics.

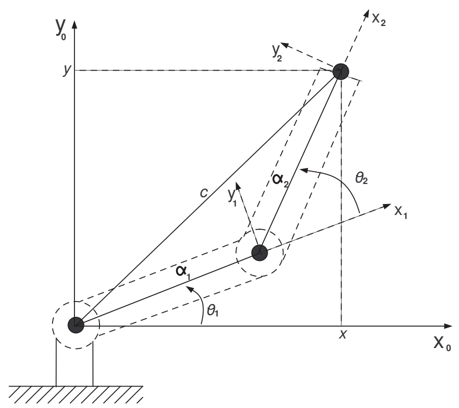

Fig. 1. Kinematic Diagram a Two-Link Planar Manipulator

3.1 Forward Kinematics

The forward kinematics is the problem of determining the Cartesian position of the end-effector of the manipulator in terms of the joint angles. This is not a straightforward task due to the presence of multiple coordinate frames, i.e., (x_0, y_0), (x_1, y_1) and (x_2, y_2). It is customary to resolve this issue by establishing a fixed coordinate system, usually referred to as the world or base frame, to which all objects, including the manipulator, can be referenced from. Mostly, this base coordinate frame is established at the origin of the manipulator. The Cartesian coordinates (x, y) of the end effector can now be expressed in this coordinate frame as [22]

$$x = \alpha_1 \cos\theta_1 + \alpha_2 \cos(\theta_1 + \theta_2) \tag{1}$$

$$y = \alpha_1 \sin\theta_1 + \alpha_2 \sin(\theta_1 + \theta_2) \tag{2}$$

where α_1 and α_2 are the lengths of the two links, respectively.

3.2 Inverse Kinematics

Inverse kinematics allows us to find the joint angles, θ_1 and θ_2, in terms of the position of the end-effector of the manipulator. The law of Cosines provides us with the following relationship for the angle θ_2

$$\cos\theta_2 = \frac{(x^2 + y^2 - \alpha_1^2 - \alpha_2^2)}{2\alpha_1\alpha_2} := D \tag{3}$$

If the desired Cartesian position coordinates (x, y) lie within the range of the manipulator's end-effector and the two links do not have to be fully extended to reach this point, then there are two different ways, i.e., elbow-up and elbow-down, in which the end-effector can reach the desired position, as shown in Figure 2. So, instead of obtaining θ_2 from the above equation, which does not distinguish between these two cases, we use the following relationships:

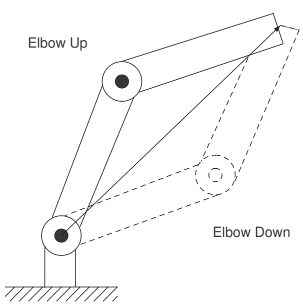

Fig. 2. Two solutions for Inverse Kinematics

$$\sin\theta_2 = \pm\sqrt{1-D^2} \tag{4}$$

$$\theta_2 = \tan^{-1}(\frac{\pm\sqrt{1-D^2}}{D}) \tag{5}$$

This way, both the elbow-up and elbow-down solutions can be obtained by choosing either the positive or the negative sign in Equation (5). Now θ_1 can be found in terms of θ_2 as follows:

$$\theta_1 = \tan^{-1}(\frac{y}{x}) - \tan^{-1}(\frac{\alpha_2\sin\theta_2}{\alpha_1+\alpha_2\cos\theta_2}) \tag{6}$$

The main contribution of this paper is the formalization of the two-link planar manipulator in higher-order logic and the formal verification of the above mentioned equations in HOL-Light. The availability of this formalization will greatly minimize the human interaction required in formal reasoning about kinematic analysis of real-world robotic applications in HOL-Light.

4 Formalization of Kinematic Analysis

We formalized the two-link manipulator, depicted in Figure 1, in the Cartesian plane as the following higher-order-logic function in HOL-Light

Definition 1: *Two Link Manipulator*
⊢ ∀ A B. tl_manipulator A B = A + B

The function tl_manipulator accepts two vectors A and B of data type ($\texttt{real}^2$), which represent the two links of the two-link manipulator, and returns their vectored sum, which is indeed the Cartesian position of the corresponding end-effector. By default, the common base frame of the two vectors is oriented at the origin. It is important to note that 2-dimensional vectors are used for the two links because of the planar nature of the mechanical system under consideration.

The two links of a general planar manipular with revolute joints can have both anticlockwise and clockwise rotations. In the context of the formal verification of the equations, presented in the previous section, it is very important to have a formal mechanism to distinguish between these two kinds of rotations. For this purpose, the following HOL-light predicate with data-type ($\texttt{real}^2 \rightarrow \texttt{real}^2 \rightarrow \texttt{real}^2 \rightarrow \texttt{bool}$) is used:

Definition 2: *Anticlockwise*
⊢ ∀ B A D. anticlockwise D (A,B) <=> 0 ≤ Im ($\frac{B-D}{A-D}$)

where Im represents the HOL-Light function that returns the imaginary part of a complex number or the y-component of a two-dimensional vector in the Cartesian plane. The predicate anticlockwise accepts three arbitrary points, A , B and D, in the Cartesian plane and returns $True$ if the angle from DB to DA is considered in the anticlockwise direction.

A very interesting property, in the context of our verification, is the following addition relationship between three anticlockwise angles:

Theorem 1: *Anticlockwise Angle Addition*

```
⊢ ∀ A B C D. ~(collinear {D, A, B}) ∧ ~(C = D) ∧
         anticlockwise D (A,C) ∧
         anticlockwise D (C,B) ∧
         anticlockwise D (A,B) ⇒
        angle  (A,D,B) = angle (A,D,C) + angle (C,D,B)
```

where `A`, `B`, `C` and `D` represent four points in the Cartesian plane. The predicate `collinear` accepts three points and returns *True* if all these three are collinear, i.e., they lie on a single straight line. Whereas, the function `angle` accepts three points `(X,Y,Z)` and returns the angle between the vectors `XY` and `ZY`.

Based on the fact that the main scope of the presented work is to build formal reasoning support for kinematic analysis in the plane geometry, we can work with a formal definition of a two dimensional angle. This choice simplifies our proofs considerably compared to the case if we had used the generic definition of an angle between n-dimensional vectors. The angle between two 2-dimensional vectors can be formally defined in terms of the angle between two n-dimensional vectors (`angle`) as follows:

Definition 3: *Two Dimensional Angle*

```
⊢ ∀ A B C. TDangle (A,B,C) =
        plus_minus (anticlockwise B (A,C)) * angle (A,B,C)
```

where the function `plus_minus` accepts a condition C and returns 1 if C is true and -1 otherwise.

Definition 4: *Plus-Minus*

```
⊢ ∀ plus_minus C = (if C then 1 else -1)
```

The the sign of the angle between two 2-dimensional vectors can be determined based on the angle of rotation between them.

4.1 Forward Kinematics

Now, we utilize the above mentioned definitions and theorem to formally verify the forward kinematic relationships, given in Equations (1) and (2). The verification of these equations is not very straightforward since all the possible scenarios for the link positions have to be considered for establishing a formal proof of the forward kinematic analysis. For example, both the links can be in the first quadrant of the Cartesian plane, as depicted in Figure 1, or one of them may lie in the first quadrant while the other one is in the third quadrant of the Cartesian plane. Our main motivation here is to verify generic theorems for kinematic analysis so that they can be used to reason about all possible scenarios.

We proceed in this direction by verifying two sub-theorems for the relationship, given in Equation (1). The first part deals with the situation when the first link lies in the upper half of the Cartesian plane while the second link and the end effector may lie any where in the Cartesian plane.

Theorem 2: *X-Component with the First Link in the Upper Half Plane*

```
⊢ ∀ A B. ∼(A = vec 0) ∧ ∼(B = vec 0) ∧
          anticlockwise (vec  0) (A,B) ∧
          anticlockwise (vec  0) (basis 1,A) ⇒
    (tl_manipulator A B)$1 =
      norm (A) * cos (vector_angle (basis 1) A) +
      norm (B) * cos (vector_angle A B + vector_angle (basis 1) A)
```

where `A` and `B` are the vectors representing the two links of the manipulator under consideration. The first two assumptions are used to avoid mathematical singularities. The third assumption ensures that the angle from the vector `OB` to `OA`, where `O` represents the origin (0,0) of the base frame, is taken in the anticlockwise direction, as the function `vec` converts `0` to its corresponding null vector. The fourth assumption, i.e., `anticlockwise (vec 0) (basis 1,A)`, ensures that the vector `A` lies in the upper half of the Cartesian plane, i.e., in the first or second quadrant, since a point $P(x, y)$ in the Cartesian plane can be considered equivalent to the point in the complex-plane with x as its real part and y as its imaginary part. It is important to note that the allowable range of any angle in the geometry theory of Hol-Light is $[0, \pi]$. The conclusion of Theorem 2 represents Equation (1) in HOL-Light, since `X$1` represents the first components of a two-dimensional vector `X`, the function `norm` accepts a vector and returns its corresponding norm or magnitude, the function `vector_angle` returns the angle between its two argument vectors and the function `basis 1` returns a vector with magnitude 1 in the x direction only and thus the angle `(vector_angle (basis 1) A)` represents the angle of the vector `A` with the x-axis. The proof of Theorem 2 was done by splitting the main goal into eight subgoals, i.e., four corresponding to the cases where the first link is in the first quadrant and the end-effector is in one of the four quadrants one by one and the four corresponding to the case where the first link is in the second quadrant and the end-effector is in one of the four quadrants one by one. Some of these cases were verified based on contradiction while the rest were primarily verified based on Theorem 1 along with vector and geometry theoretic reasoning.

Now we consider the second case when the first link lies in the lower half of the Cartesian plane while the end effector may lie any where in the Cartesian plane. The corresponding HOL-Light theorem can be expressed as follows:

Theorem 3: *X-Component with the First Link in the Lower Half Plane*

```
⊢ ∀ A B. ∼(A = vec 0) ∧ ∼(B = vec 0) ∧
          anticlockwise (vec 0) (A,B) ∧
          ∼(anticlockwise (vec 0) (basis 1,A)) ⇒
    (tl_manipulator A B)$1 =
      norm (A) * cos (vector_angle (basis 1) A) +
      norm (B) * cos (vector_angle A B - vector_angle (basis 1) A)
```

The only different assumption in this case is `∼(anticlockwise (vec 0) (basis 1,A))`, which ensures that the vector `A` lies in the lower half of the Cartesian plane, i.e., in the third or fourth quadrant. Now, Theorems 2 and 3 along with their counterparts with the assumption `~(anticlockwise (vec 0) (A,B))` can be utilized to verify the following theorem that covers all the possible cases for forward kinematic analysis of the x-component of the two-link manipulator (Equation 1)

Theorem 4: *X-Component of the Two-Link Manipulator*

```
⊢ ∀ A B. ~(A = vec 0) ∧ ~(B = vec 0) ⇒
    (tl_manipulator A B)$1 =
      norm (A) * cos (TDangle (basis 1,vec 0,A)) +
      norm (B) * cos (TDangle (A,vec 0,B) +
                      TDangle (basis 1,vec 0,A))
```

In a similar way, we also formally verified the forward kinematic equation for the y-component of the two-link manipulator (Equation 2) as follows:

Theorem 5: *Y-Component of the Two-Link Manipulator*

```
⊢ ∀ A B. ~(A = vec 0) ∧ ~(B = vec 0) ⇒
    (tl_manipulator A B)$2 =
      norm (A) * sin (TDangle (basis 1,vec 0,A)) +
      norm (B) * sin (TDangle (A,vec 0,B) +
                      TDangle (basis 1,vec 0,A))
```

A distinguishing characteristic of Theorems 4 and 5 is the presence of the function `TDangle`, which provides us with the exact information about the sign of the corresponding angle based on the placement of the two links in the Cartesian plane. This fact plays a very critical role in obtaining the right set of analysis assumptions, as will be seen in Section 5 of this paper.

4.2 Inverse Kinematics

In this subsection, we formally verify the expressions for joint angles in terms of the Cartesian position of the end-effector. The expression for the angle θ_2, which is the phase angle of the second link with respect to the reference frame of the first link of the two-link manipulator, as shown of Figure 1, can be verified in three main steps corresponding to Equations (3), (4) and (5). We verified the expression, given in Equation (3), as the following theorem,

Theorem 6: *Cos* θ_2

```
⊢ ∀ A B. ~(A = vec 0) ∧ ~(B = vec 0) ∧ ~(A + B = vec 0) ⇒
         cos (vector_angle A B) =
            ((((tl_manipulator A B)$1) pow 2) +
             (((tl_manipulator A B)$2) pow 2) - (norm (A) pow 2) -
                 (norm (B) pow 2)) / (2 * norm (A) * norm (B))
```

where `A` and `B` are the vectors representing the two links of the manipulator under consideration. Besides the vectors `A` and `B` being non-null, their sum is also assumed to be non-null in this theorem. This assumption is used to make sure that we do not have the case where both the vectors are equal in magnitude but opposite in direction. Based on our formal definition of the two link manipulator, `(tl_manilpulator A B)$1` represents the x-coordinate and `(tl_manilpulator A B)$2` represents the y-coordinate of the end-effector. The function `pow` is used for real power in HOL-Light. The proof of the above theorem was primarily based on the Law of Cosines, which is available in the geometry theory of HOL-Light.

Similarly, the expression of Equation (4) can be verified as the following theorem:

Theorem 7: *Sin* θ_2

```
⊢ ∀ A  B. sin (vector_angle  A  B) =
         sqrt(1 - cos (vector_angle  A  B) pow 2)
```

where `sqrt` represents the HOL-Light function for square root. The proof of this theorem was also primarily based on some geometry theoretic reasoning and the existing theorems of the HOL-Light geometry theory were very helpful in this regard. Theorems 6 and 7 can now be used to verify our final results, i.e. Equation (5), as the following theorem.

Theorem 8: θ_2

```
⊢ ∀ A B. ∼(A = vec  0) ∧ ∼(B = vec  0) ∧ ∼(A + B = vec 0) ⇒
         vector_angle  A  B =
           atn ((sqrt (1 - (cos (vector_angle A B) pow 2))) /
               cos  (vector_angle  A B))
```

where `atn` represent the arc-tangent function in HOL-Light. The proof scripts for the definitions and theorems, presented in this section, is available at [3] and is composed of approximately 15,000 lines of code, which took about 700 man-hours of development time by a new HOL-Light user. The development time includes the time spent to learn the art of formal reasoning and to understand the definitions and theorems available in the multivariate and geometry theories of HOL-Light. We found the generic nature of the geometry theory, formalized in HOL-Light, very useful since most of the commonly used results can be found there and thus directly used. A significant portion of our proof script is composed of mapping between Cartesian to Polar coordinate system, which was required because of the referencing to the base frame for both links of the manipulator. These mappings had to be interactively done because of the different parameters and thus conditions involved. Moreover, in all the proofs we had to handle the placement of the links in each quadrant separately, which also required a considerable amount of human guidance.

The main benefit of the formalization, presented in this section, is that it can be built upon to model robotic structure and perform the kinematic analysis or formalize other kinematic analysis related foundations, which will in turn enhance the capabilities of the proposed formal kinematic analysis approach. To demonstrate the utilization and practical effectiveness of our results, we utilize them next to conduct the formal kinematic analysis of a biped robot.

5 Application: Biped Robot

A biped robot [12] can be simply defined as a two-legged walking robot. The biped robots usually exhibit human-like mobility and have found to be more efficient than the conventional wheeled robots for maneuvering fields with ladders, stairs, and uneven surfaces. The design and analysis of efficient biped robotic systems has attracted the attention of many researchers and it has been used as a classical case study for many domains in robotics, which is the main reason why we have also chosen to use it as an application to demonstrate the effectiveness of the proposed formal kinematic analysis approach.

The biped robot, depicted in Figure 3, consists of five links, namely the torso ($l3$), and the upper legs (thighs), i.e, $l2$ and $l4$, and the lower legs, i.e., $l1$ and $l5$. These links are connected via four rotating joints (two hip and two knee), which are considered to be friction free and each one is driven by an independent DC motor. The two hip joints can be considered as one joint as the effect of forces is ignored while conducting the kinematic analysis.

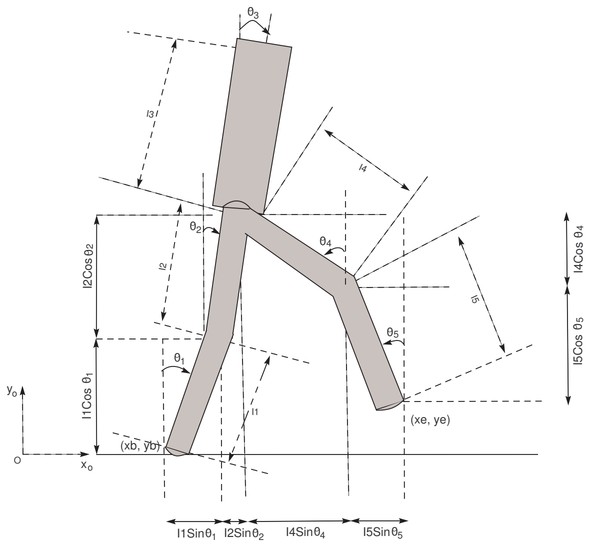

Fig. 3. Biped Robot [12]

The first step in the proposed formal kinematic analysis approach is to construct a formal model of the given system in higher-order logic. The definition of the two-link manipulator plays a vital role in this regard as it can be used to model most of the robot manipulators. The given biped robot can be formalized as a two-link planar manipulator where each link is itself a two-link planar manipulator, since both of the legs always move in the same plane:

Definition 5: *Biped Robot*

```
⊢ ∀ A B C D. biped A B C D =
        tl_manipulator (tl_manipulator A B) (tl_manipulator C D)
```

It is important to note that only four out of the five links are considered in the above definition. The reason being that these four links are the ones that can alter the dexterity and precision of the biped robot. Moreover, the point (x_b, y_b) of Figure 3 has been taken as the origin $(0, 0)$ in our formalization.

The next step in the proposed kinematic analysis approach is to formalize the properties of interest as higher-order-logic proof goals. In the case of the biped robot, we are interested in verifying the forward kinematic equations:

$$x_e = x_b + l_1 \sin\theta_1 + l_2 \sin\theta_2 + l_4 \sin\theta_4 + l_5 \sin\theta_5 \tag{7}$$

$$y_e = y_b + l_1 \cos\theta_1 + l_2 \cos\theta_2 - l_4 \cos\theta_4 - l_5 \cos\theta_5 \tag{8}$$

corresponding to the x and y components of leg $l5$ of the biped robot, which have been verified using paper-and-pencil proof method in [12]. Besides formalizing these equations, we also need to mention the required assumptions in the proof goals but it is often the case that all the assumptions are not known upfront and we only add the obvious ones. However, the subgoals generated during the proof process provide very useful hints for identifying any missing assumptions, which may be added to the goal after confirmation from the mechanical designers. This way, we verified Equation (7) as the following theorem.

Theorem 9: *x-component of the Biped Robot*

```
⊢ ∀ l1 l2 l4 l5.
∼(l1 = vec 0) ∧ ∼(l2 = vec 0) ∧ ∼(l4 = vec 0) ∧ ∼(l5 = vec 0) ∧
  anticlockwise (vec 0) (l1,l2) ∧ anticlockwise (vec 0) (l5,l4) ∧
        anticlockwise (vec 0) (basis 1,l1) ∧
        anticlockwise (vec 0) (l1,basis 2) ∧
        anticlockwise (vec 0) (basis 1,l2) ∧
        anticlockwise (vec 0) (l2,basis 2) ∧
        anticlockwise (vec 0) (basis 1,l5) ∧
        anticlockwise (vec 0) (basis 2,l5) ∧
        anticlockwise (vec 0) (basis 2,l4) ∧
        ∼ collinear {basis 1, vec 0, l5} ∧
        ∼ collinear {basis 1, vec 0, l2} ∧
        ∼ collinear {basis 2, vec 0, l4} ⇒
            (biped l1 l2 l4 l5)$1 =
              norm (l1) * sin (vector_angle (basis 2) l1) +
              norm (l2) * sin (vector_angle (basis 2) l2) -
              norm (l4) * sin (vector_angle (basis 2) l4) -
              norm (l5) * sin (vector_angle (basis 2) l5)
```

where basis 2 represents the y-axis of the Cartesian plane. The first four assumptions assure that all of the links must have non-zero lengths. The next two provide the angle measuring conventions between the links of each leg while the next eight assumptions provide the orientations of the angles, given in Figure 3, under which we are interested in analyzing the biped robot. For example, the assumptions `anticlockwise (vec 0) (basis 1,l1)` and `anticlockwise (vec 0) (l1,basis 2)` ensure that the angle θ_1 of leg $l1$ of the robot is considered in the clockwise direction from the x-axis of the base frame and in the anticlockwise direction with the y-axis of the base frame, or in other words that θ_1 lies in the first quadrant. All of the above mentioned assumptions are obvious and were used while defining the proof goal. While the next three, involving the function `collinear` are not so straightforward and thus were not known before the formal verification of the above theorem. These assumptions assure that the legs $l5$ and $l2$ cannot completely align with the x-axis of the base frame and the leg $l4$ cannot align with the y-axis of the base frame. These three assumptions were identified during the formal verification process of Theorem 9 in HOL-Light. These assumptions constitute an essential requirement for Equation (7) to hold and interestingly were not mentioned in the paper-and-pencil based proof of that equation given in [12]. The conclusion of Theorem 9 represents the statement of Equation (7) but instead of having all positive terms added together, as is the case in Equation (7), has the addition of two positive and two negative terms. Upon noticing this difference, we double checked our formalization and assumptions but everything was found to be consistent with the kinematic diagram of the biped robot, given in Figure (3), which was also the starting point of the paper-and-pencil proof method based kinematic analysis of the biped robot from where Equation (7) was verified [12]. Upon further investigation, we found Equation (7) to be erroneous. The problem may be an outcome of a human error during the paper-and-pencil based analysis or it may have occurred because of an incorrect swapping of the signs of the terms of Equations (7) and (8) in the original paper [12] as we also found a discrepancy in Equation (8), which has been correctly verified under the same assumptions as follows:

Theorem 10: *y-component of the Biped Robot*

```
⊢ ∀ l1 l2 l4 l5.
~(l1 = vec 0) ∧ ~(l2 = vec 0) ∧ ~(l4 = vec 0) ∧ ~(l5 = vec 0) ∧
  anticlockwise (vec 0) (l1,l2) ∧ anticlockwise (vec 0) (l5,l4) ∧
        anticlockwise (vec 0) (basis 1,l1) ∧
        anticlockwise (vec 0) (l1,basis 2) ∧
        anticlockwise (vec 0) (basis 1,l2) ∧
        anticlockwise (vec 0) (l2,basis 2) ∧
        anticlockwise (vec 0) (basis 1,l5) ∧
        anticlockwise (vec 0) (basis 2,l5) ∧
        anticlockwise (vec 0) (basis 2,l4) ∧
        ~ collinear {basis 1, vec 0, l5} ∧
        ~ collinear {basis 1, vec 0, l2} ∧
        ~ collinear {basis 2, vec 0, l4} ⇒
            (biped  l1  l2  l4  l5)$2 =
              norm (l1) * cos (vector_angle (basis 2) l1) +
```

```
norm (l2) * cos (vector_angle (basis 2) l2) +
norm (l4) * cos (vector_angle (basis 2) l4) +
norm (l5) * cos (vector_angle (basis 2) l5)
```

Nonetheless, the identification of the problem identifies the dire need of a sound analysis technique in the domain of kinematic analysis of robotic applications, which is quite cumbersome and thus prone to human errors if done using paper-and-pencil proof method because of the consideration of all the possible scenarios. The proofs of Theorems 9 and 10 were very straightforward and primarily based on Theorems 5 and 6 and the proof script consists of approximately 200 lines of code [3], which clearly indicates the usefulness of our work in the formal kinematic analysis of real-world applications.

It is important to note that, besides the identification of a bug in the paper-and-pencil based proof method of kinematic analysis of the biped robot, another distinguishing feature of the above theorems, when compared with the corresponding equations verified by paper-and-pencil proof methods [12], is the exhaustive set of assumptions that accompany them. Paper-and-pencil proof methods, simulation and numerical method based approach cannot ascertain the explicit availability of all of the required assumptions for the analysis even though failing to abide by any one of these assumptions may lead to erroneous system designs, which in turn even result in disastrous consequences in the case of safety-critical systems. On the other hand, given the intrinsic soundness of theorem proving, all the required assumptions are always guaranteed to be available along with the formally verified proofs.

6 Conclusion

This paper advocates the usage of higher-order-logic theorem proving for kinematic analysis, which is an essential design step for all robotic manipulators. Due to the high expressiveness of the underlying logic, we can formally model the geometrical relationships between the mechanical links and joints in their true form, i.e., without compromising on the precision of the model. Similarly, the inherent soundness of theorem proving guarantees correctness of analysis and ensures the availability of all pre-conditions of the analysis as assumptions of the formally verified theorems. To the best of our knowledge, these features are not shared by any other existing computer-based kinematic analysis technique and thus the proposed approach can be very useful for the kinematic analysis of safety-critical robots.

The main challenge in the proposed approach is the enormous amount of user intervention required due to the undecidable nature of the logic. We propose to overcome this limitation by formalizing kinematic analysis foundations and verifying their associated properties. As a first step towards this direction, this paper presents the formalization of forward and inverse kinematic analysis of a two-link planar manipulator, which can be built upon to model many practically used robotic manipulators. Based on this work, we are able to conduct the formal kinematic analysis of a biped robot in a very straightforward way. The formal analysis highlighted some of the key missing assumptions in the corresponding paper-and-pencil based analysis of the same problem [12], which clearly indicates the dire need of using sound methods for kinematic analysis and usefulness of the ideas presented in this paper.

This paper opens the doors towards a novel and promising usage of theorem proving. The formal kinematic analysis of many other safety-critical robots can be performed. A classical example would be the widely used Selective Compliant Assembly Robot Arm (SCARA) [4]. Formal reasoning support for other motion related parameters, such as displacement, rotation, speed, velocity and acceleration, may also be developed by using our results along with the topological and analytic foundations for vectors available in Hol-Light. By extending our coordinate frame from 2D to 3D, we are also working on the formal verification of the Denavit Hartenberg (DH) parameters [5], which are based on the convention of attaching the coordinate frame to spatial linkages. This verification will pave the way for conducting the formal kinematic analysis for a much wider range of robotic applications.

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