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Analysis Techniques for Fractional Order Systems: A Survey

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Abstract. Fractional calculus, which deals with the integration and differentiation of non-integer order, is widely being used these days to mathematically model many engineering and scientific systems ranging from microbiological processes to astronomical images. The efficient and accurate analysis of such systems, usually referred to as the fractional order systems, has become very important and a number of analysis techniques have been recently proposed. This paper provides a brief survey about these fractional order system analysis techniques while highlighting upon their key strengths and drawbacks. In particular, we evaluate each analysis technique with respect to its expressiveness, accuracy and scalability. We also propose a hybrid analysis technique, which is more efficient and reliable, for the analysis of fractional order systems.

Keywords: Fractional Calculus, Theorem Proving, Computer Algebra Systems, Computer Simulations
PACS: 03F03, 26A33, 65Y04

INTRODUCTION

In 1695, L'Hôpital asked Leibnitz about the notation $\frac{d^n y}{dx^n}$: "*What if n is* $\frac{1}{2}$". Leibnitiz [1] prophesied in his letter to L'Hôpital,"*...Thus it follows that* $d^{\frac{1}{2}}x$ *will be equal to* $x\sqrt{dx : x}$. *This is an apparent paradox from which, one day, useful consequences can be drawn* ...". Leibnitz [2] continued with this idea and discussed with Johann Bernouli about the derivatives of 'general orders'. After three years, in his correspondence with John Wallis, Leibnitz [3] discussed the ways for using fractional derivatives in finding the infinite product for $\frac{1}{2}\pi$. Euler, a great mathematician and physicist, also contributed in the development of fractional calculus. In 1730, he proposed in his dissertation, an integral expression for the Gamma function which generalizes factorial over non integer numbers. The Gamma function plays the same role in the fractional calculus as factorial in the integer order calculus [4].

The accuracy and robustness of control systems are becoming imperative these days. The dynamic nature of control systems requires them to be modeled using the fractional calculus. In control engineering the concept of the fractional operations is mostly used in fractional system identification [5] and biomimetic (bionics) control [6]. The other applications of fractional calculus are in Signal processing [7], Image Processing [8], Electromagnetic Theory [9, 10],Communication [11] and Biology [12].

Now, we provide the two most commonly used definitions of fractional calculus and discuss the choice of their usage in different applications. This information is used in the next section to compare capabilities of the available computer based fractional order system analysis techniques.

Riemann-Liouville (RL) Definition:

$$J_a^v f(x) = \frac{1}{\Gamma(v)} \int_a^x (x-t)^{v-1} f(t) dt \tag{1}$$

Where $J_a^v f(x)$ represents fractional integration with order v and lower integration limit a. $a = 0$ gives the Riemann definition and $a = -\infty$ gives the Liouville definition of fractional integration [13]. Γ in the above definition denotes the Gamma function which is defined using the well-known improper integral as follows:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{2}$$

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for $z > 0$.
The fractional differentiation is given as follows:

$$D^{v} f(x) = (\frac{d}{dx})^{m} J_{a}^{m-v} f(x) \tag{3}$$

where m represents the ceiling of v, i.e., $\lceil v \rceil$.
Grünwald-Letnikov (GL) Definition:

$${}_{c}D_{x}^{v} f(x) = \lim_{h \to 0} h^{-v} \sum_{k=0}^{[\frac{x-c}{h}]} (-1)^{k} \binom{v}{k} f(x-kh) \tag{4}$$

Grünwald-Letnikov definition caters for both fractional differentiation and integration in the same expression, as positive values of v give fractional differentiation and negative values of v give fractional integration. Here, $\binom{v}{k}$ represents the binomial coefficient, which is described in terms of the Gamma function.

The Riemann-Liouville definition provides a way to find analytical solutions while Grünwald-Letnikov definition facilitates the numerical computation of solutions. There are two motivations of using the Riemann-Liouville definition: Firstly, it is widely used in the modeling and analysis of engineering fractional order systems [14], Secondly, the analysis carried out in this way is purely analytical and hence free from any kind of approximations. On the other hand, Grünwald-Letnikov definition is more suitable for numerical analysis based methods and thus provides approximate solutions. A number of fractional order system analysis techniques have been developed using these definitions and some of the widely used ones are described in the next section.

ANALYSIS TECHNIQUES

Paper-and-Pencil proof Method: The main idea in this classical technique is to construct a mathematical model of the given fractional order system on paper using the Riemann-Liouville (RL) definition. This model is then used to verify that the system exhibits the desired properties using mathematical reasoning on a paper. Traditionally, the analysis of fractional calculus based models has been done using paper-and-pencil proof methods [14, 9, 15]. However, considering the complexity of present age engineering and scientific systems, such kind of analysis is notoriously difficult, if not impossible, and is quite error prone. Many examples of erroneous paper-and-pencil based proofs are available in open literature, a recent one can be found in [16] and its identification and correction is reported in [17].
Computer Simulations: The availability of high speed computers attracted the attention of researchers working in the field of fractional order system analysis to perform simulation based analysis using numerical algorithms [18]. The main idea of simulation based methods is to construct a discretized system model and then simulate the output of the system using different input patterns. Some examples include, chaos in fractional order volta systems [19], fractional PI^{α} controllers [20] and motion planning of redundant and hyper-redundant manipulators [21]. Most of the numerical algorithms are based either on Grünwald-Letnikov definition [18] or on power series expansion (PSE) method [19]. Both of them cannot provide valuable results due to involvement of infinite summations in case of Grünwald-Letnikov definition and huge memory requirements in case of the power series expansion method. Similarly, the computation of the Gamma function $\Gamma(x)$ for large values of x is not possible in such numerical computation software packages. For example, MATLAB [22] returns `7.26e306` as the approximated value computed for $x = 171$ and returns `Inf` for all values beyond $x = 171$.
Computer Algebra Systems: The computer algebra systems [23] are becoming popular for the fractional order system analysis [24]. In computer algebra systems, the mathematical computations are done using symbolic algorithms, and hence they are better than simulation based analysis in terms of precision. But the simplification performed by computer algebra systems are not reliable [25] due to their limitations of dealing with side conditions. Another limitation of computer algebra systems is the uncertain simplification of singular expressions particularly in case of the Gamma function [26], which are frequently used in fractional calculus due to usage of improper integrals. Another source of inaccuracy in computer algebra systems is the presence of unverified huge symbolic manipulation algorithms in their core, which are quite likely to contain bugs [25].
Theorem Proving: Theorem proving [27] or automatic deduction, which is one of the most developed research area in formal methods, is concerned with the construction of mathematical theorems using a computer program. The main idea behind the theorem proving based formal analysis is to mathematically model the given system in an appropriate

logic and then the properties of interest are verified using computer based formal reasoning. Using higher-order logic theorem proving for modeling the system behaviors makes the analysis very flexible as it allows the formal verification of any system that can be expressed mathematically. The core of theorem provers usually consists of some well-known axioms and primitive inference rules. The theorem proving based verification assures the soundness as every new theorem must be created from these basic axioms and primitive inference rules or any other already proved theorems. Recently, Siddique and Hasan [28] presented the first formal framework for the analysis of fractional order systems. This framework is based on higher-order-logic formalization of the Riemann-Liouville (RL) definition and effectiveness of the approach was demonstrated by some small case studies of resistoductance and fractional order differentiator and integrator. The main limitation of this approach is the scalability and availability of mathematical infrastructure involved in complex fractional order systems, such as fractional waveguides.

DISCUSSIONS

Due to the extensive usage of engineering and scientific systems in safety-critical domains, such as medicine and transportation, the accuracy of system analysis has become extremely important. Fractional order systems are natural extensions of integer order systems and provide more accurate and robust modeling capabilities to engineers and scientists. This fact led to the use of fractional order systems in the safety-critical domains (e.g., cardiac tissue electrode interface [29] which is modeled and analyzed using fractional calculus). Considering the above mentioned facts, it is very important to build a framework for the analysis of fractional order systems which is accurate and scalable. The comparison of all the existing fractional order system analysis techniques is given in Table 1. These techniques are evaluated according to their expressiveness, scalability, accuracy, availability of fractional order system fundamentals (Gamma function and improper integrals) and the possibility of the automation of the analysis.

Paper-and-Pencil proof based analysis is accurate for small systems but for larger systems it is very difficult to maintain the correctness of the analysis. Another limitation of this technique is that automation is not possible and analysis can only be carried out manually by hand which is very tedious for complex fractional order systems. Computer simulations are expressive but the main limitations is the scalability and accuracy. For complex fractional order systems it is impossible to simulate all the cases and the numerical nature of the analysis results in the inaccurate output behaviors. Computer algebra systems which are expressive, scalable and automatic also have problem with their accuracy due to limitation of dealing with side conditions and unverified symbolic algorithms. Theorem proving is accurate, scalable and expressive but the availability of fractional order system fundamentals and automation of the analysis are questioned for the large and complex systems. Currently, the available mathematical infrastructure in theorem proving is not mature enough to handle complex real-world fractional order systems. For example, fractional laplace transform and other definitions of fractional calculus (Caputo derivative) are not available which limits its use for the analysis of complex fractional order systems.

In the light of above discussion, we recommend a hybrid framework for the analysis of fractional order systems. Here, hybrid means a framework which is based on a higher-order-logic theorem prover and a computer algebra system [30]. Hybrid framework would greatly facilitate the analysis of fractional order systems because theorem proving assures accuracy and soundness whereas computer algebra system are very efficient and mature enough for the fractional order system analysis. In general, the flow of the proposed hybrid analysis would be as follows: First step is to construct a formal model of given fractional order system in higher-order logic. Second step is to describe the systems properties as theorems in higher-order logic. Last step is to formally verify the theorems using the available infrastructure in higher-order logic theorem proving if possible otherwise some parts of the theorem can be solved in computer algebra system. The major challenge in developing such a framework would be to create the automatic link between the two frameworks such that the computer algebra system may be invoked from within the theorem prover for verifying any given subgoal specified in higher-order logic.

CONCLUSIONS

In this paper, we presented a survey of the analysis techniques for the fractional order systems. We also presented a brief overview of each analysis technique and highlights its strengths and weaknesses. At the end, we recommended a hybrid (theorem proving and computer algebra systems based) framework for the analysis of fractional order systems. Currently, we are working on the development of a hybrid framework which is more powerful than the existing theorem proving based framework [28] and can be utilized for the analysis of more complex real-world fractional order systems.

TABLE 1. Comparison of Analysis Techniques for Fractional Order Systems

	Paper-and-Pencil Proof	Simulation	Computer Algebra System	Theorem Proving
Expressiveness	Yes	Yes	Yes	Yes
Scalability	No	No	Yes	Yes
Accuracy	Yes-**?**	No	No	Yes
FOS Fundamentals	Yes	Yes	Yes	Yes-**?**
Automation	No	Yes	Yes	Yes-**?**

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